AN EVOLUTION OPERATOR SOLUTION FOR A NONLINEAR BEAM EQUATION

DISSERTATION

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A nonlinear partial differential equation, motivated by the transverse vibration of a beam, is shown to have a unique solution. The existence theory, which is in the setting of semigroups and evolution operators, is a composite and synthesis of theorems of Kato. The formulation of the problem and the verification that the formulation leads to a solution are new.
The introductory chapter provides background on the topic generally. Chapter 2 provides detailed formulations for the constant coefficient case. Chapter 3 describes nonautonomous cases. The most general theorem is presented here. In Chapter 4, a more general case is considered. Namely, Kelvin-Voigt damping with a coefficient which depends on the solution. This introduces a nonlinearity to the problem which makes it of the form frequently called quasilinear. This is a stronger form of nonlinearity than semilinear. Results of a numerical example are presented.

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## *Preface*

My research proposal was based on my search for an unsolved problem concerning the dynamics of large space structures. My strategy was to identify some feature of the dynamics for which adequate mathematical tools were not available. In my reading, on several occasions, I encountered phrases about nonlinearities not being adequately modelled. I determined that some of the simple traditional models for beam behavior did not have nonlinear counterparts. That is to say, while many sophisticated models had been developed, some rather straightforward models apparently had not. I was curious whether the mathematics was available for such models. Finding none, and obtaining the concurrence of my committee, I started my research.

I do not know how well my model will serve for analysis of space structures. I do know that the analysis of mathematical tools has been fascinating and enjoyable.

I would like to extend many thanks to the courteous and professional staff of the AFIT Library. They have efficiently handled countless Interlibrary Loan requests and have helped in a variety of local searches. I thank them all.

I would also like to mention that when my proposed research topic was still too broad, Dr. Bagley posed several questions which were very effective in helping me to focus more rapidly on a specific equation. I thank him.

Many thanks are also due to Dr. Lair, who answered a variety of technical questions and pointed out several refinements. His countless hours, in my behalf, are greatly appreciated.

Finally, I thank Dr. Quinn. He has been a constant source of encouragement and guidance. He was always eager and helpful in guiding me in the pursuit of my research. He went to great effort to guide my efforts as a researcher so that they would be most effective. Yet, he was careful not to redirect me along any lines other than those I wished to pursue. He taught me to do research.

Carl Edwin Crockett

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## *Abstract*

A nonlinear partial differential equation, motivated by the transverse vibration of a beam, is shown to have a unique solution. The existence theory, which is in the setting of semigroups and evolution operators, is a composite and synthesis of theorems of Kato. The formulation of the problem and the verification that the formulation leads to a solution are new.

The introductory chapter provides background on the topic generally. Chapter 2 provides detailed formulations for the constant coefficient case Chapter 3 describes nonautonomous cases. The most general theorem is presented here. In Chapter 4, a more general case is considered. Namely, Kelvin-Voigt damping with a coefficient which depends on the solution. This introduces a nonlinearity to the problem which makes it of the form frequently called quasilinear. This is a stronger form of nonlinearity than semilinear. Results of a numerical example are presented.

# AN EVOLUTION OPERATOR SOLUTION FOR A NONLINEAR BEAM EQUATION

## I. Introduction

The central problem of this dissertation concerns a nonlinear partial differential equation for the modelling of the transverse vibration of a beam. The particular form of nonlinearity that arises gives the equation a form that is frequently referred to as quasilinear. This is a stronger nonlinearity than the form known as semilinear. The equation is shown to have a unique solution. The applicable existence theory is a composition of some theorems of Kato. The formulation of the problem and the verification that the formulation leads to a solution are new.

This introductory chapter provides background on the topic generally. Chapter 2 provides detailed formulations for the constant coefficient cases. Chapters 3 and 4 provide the generalizations.

### *1.1 Some general background*

Partial differential equations are an essential element in the mathematical modelling of the physical world. The topics that arise in the study of natural phenomena with partial differential equation models are numerous and varied. Indeed, the fact that two physical phenomena are very similar does not guarantee any particular connection between the corresponding mathematical models. In fact, two models of the same behavior, with one slightly more sophisiticated than the other, can lead to entirely distinct mathematical entities. Even changes which appear to be very minor or superficial can radically alter the nature of the solution, and even whether there is a solution. In cases where a solution still exists, it may be necessary to use entirely different methods to find it (or them).

In the classical theory of partial differential equations the three basic second order forms are

1. the heat equation $u_t = u_{xx}$

2. the wave equation $u_{tt} = u_{xx}$

3. Laplace's equation $u_{tt} = -u_{xx}$.

It is well known that these have quite distinct solution characteristics. The same holds true for higher order equations. When problems are formulated abstractly, it remains true that small changes in appearance can lead to significant changes in solution properties.

The equations above have been given names to distinguish their types. The heat equation is known as an example of a parabolic equation. It has a unique real characteristic curve. The wave equation is called hyperbolic. It has two distinct characteristic curves. The Laplace equation is elliptic and has no real characteristic curves. The classifications are used to distinguish equations whose solutions have certain distinctive behavior.

As an example of a model refinement which may not look too different, Friedman and Hu [17:pp 249,252] discuss an equation referred to as the hyperbolic heat equation. It arises when a slight modification to a basic mathematical assumption avoids the physically unrealistic property, of the usual model, that global temperature changes instantaneously. The change is to replace the Fourier Law $q = -k\theta_x$ with $q + \tau q_t = -k\theta_x$ for some time delay $\tau$, probably quite small. The resulting equation is

$$\tau\theta_{tt} + \theta_t - \theta_{xx} = 0. \tag{1}$$

In the case of abstract equations the affect of a negative sign or of a change in the order of the equation can be pointed out by observing that the abstract equations

$$\begin{aligned} u_t &= u_{xx} \\ u_{tt} &= u_{xx} \\ u_{tt} &= -u_{xxxx} \end{aligned} \tag{2}$$

all have classical solutions. But, in their abstract formulation with homogeneous Dirichlet boundary conditions, none of the equations

$$u_t = -u_{xx}$$

$$
\begin{aligned}
u_{tt} &= -u_{xx} \\
u_{tt} &= u_{xxxx}
\end{aligned}
\tag{3}
$$

has a nontrivial classical solution.

The terms parabolic and hyperbolic do not have the same meaning in the abstract theory as they do for classical second order equations. (The term elliptic is not used for abstract equations.) An abstract equation of the form

$$
u_t = Au \tag{4}
$$

is called parabolic if the operator $A$ is the generator of an analytic semigroup. It is called hyperbolic if $A$ is the generator of only a $C_0$ semigroup (see Definition 2).

The present work addresses an Euler-Bernoulli beam equation with Kelvin-Voigt damping. The abstract formulation results in an equation which is hyperbolic. Linear and nonlinear versions will be considered as well as autonomous and nonautonomous versions.

### *1.2 An introduction to the basic equation*

A fundamental element of most large structures is the beam. Over a period of many years, as the need for more and more accurate modelling has become apparent, the mathematical equations used to describe beam behavior have become more and more sophisticated. Similarly, more advanced mathematical tools are used in the development and analysis of such equations. Herein, attention will be focused on equations for the transverse vibration of a beam.

In particular, an equation with nonlinear damping will be considered. This will be preceeded by the treatment of simpler equations in order to build up the appropriate mathematical machinery. Existence and uniqueness of a solution, in the simple cases, will be established from the viewpoint of semigroup theory. The more complex cases will require the mild generalization of a semigroup known as an evolution system.

In this section a review of an early model of beam motion will be presented. In the next chapter the model will be considered from a mathematically sophisticated (*ie*

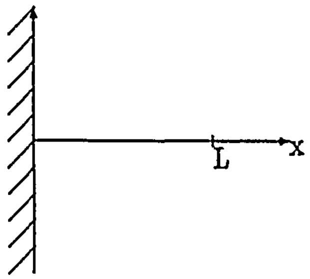



Figure 1. Cantilevered beam

semigroups) point of view. This will provide a point of departure for the investigation of more complex and potentially more precise mathematical models of beam behavior. Development of the mathematics related to these models is the purpose of the present work. It is stated at the outset that while beam equations motivated this mathematical problem, the engineering is left to others. That is to say, while the equations developed may be of engineering value, it will not be proved here. The current work is limited to establishing of the validity of certain mathematical operations.

### *1.2.1 The traditional starting place*

Consider the cantilevered beam. This is a beam which is fixed at one end, as to a wall, and free at the other end (see Figure 1). Assume a thin beam whose density and other physical characteristics are uniform. It is customary to establish a coordinate system, assumed inertial, whose origin is at the intersection of the beam's centerline and the wall. The $x$-axis is chosen along the beam's centerline. It is supposed that the centerline is straight and represents the natural, rest, or unperturbed position of the beam. Let $L$ be the length of the beam.

Vertical motion of each point $x$ along the beam, with vertical displacement at time $t$ represented by $u(t,x)$, is considered. Notice that the physical configuration requires (for consistency) the following mathematical assumptions:

$$u(t,0) = 0 \tag{5}$$

$$u_x(t,0) = 0 \tag{6}$$

$$u_{xx}(t,L) = 0 \tag{7}$$

$$u_{xxx}(t, L) = 0. \tag{8}$$

(Subscripts indicate partial differentiation with respect to the variable appearing as the subscript.) The physical interpretations of the boundary conditions are as follows: (5) says that the left end of the beam is held in a fixed position, (6) says that the beam centerline is perpendicular to the wall, (7) says that there is no curvature at the right end of the beam, and (8) says that the curvature at the right end is not changing (*ie* there is no external torque).

A large variety of physical situations can be described with only slight changes in (5)-(8). For example if both ends were free, conditions like (7) and (8) would apply at the left end. If a control mechanism were attached at the right end, then conditions (7) and (8) would have nonzero right hand sides. Clearly there are many other physical situations described by closely related sets of conditions. Of course, $u(t, x)$ must be sufficiently differentiable for the boundary conditions to make sense. An assumption to this affect is consistent with typical physical situations.

The simplest model for the transverse motion of a beam is called the Euler-Bernoulli beam model. It has been in use for over one hundred years. (See Russell [67:pp 177-216] for an excellent historical review of models.) The following assumptions are made:

1. The density of the beam, $m(x)$, is known and appropriately smooth. It is assumed to be constant with respect to $t$.
2. Vertical displacements are small compared to the length of the beam and any horizontal component of motion is neglected.
3. Young's modulus, $E$, (an experimentally determined, material dependent quantity that relates the amount of stretching to the amount of force applied, see Kolsky [37:pg 9] for example) and the area moment of inertia along the bending axis, $I$, are known and sufficiently smooth (often taken as constant). They are assumed to be constant with respect to time.
4. Mechanical energy is conserved.

Details of the derivation are provided in Appendix A. The resulting equation is

$$u_{tt} + (\alpha u_{xx})_{xx} = 0; \quad 0 \leq x \leq L; \quad 0 \leq t \tag{9}$$

where $\alpha = \frac{EI}{m}$.

This is the simplest mathematical model for a beam. Transverse vibration is allowed and elasticitv of bending is considered. Longitudinal stretching and vibration are ignored, as are time dependent changes in material properties. Also ignored are changes in material properties due to the history and current state of the beams position and motion. Damping is not modelled and other imperfections are sure to be present. Nevertheless, some starting place is needed and the Euler-Bernoulli model is the usual place. It provides a specific equation for use in demonstrating the mathematical tools to be used in further analysis.

### *1.2.2 Proposed generalizations and analysis of the equation*

The linear partial differential equation above is reduced to a system which is first order in time. Then an appropriate Banach space is chosen and the system is formulated as an abstract Cauchy problem. Existence and uniqueness of a solution is then established from standard theorems. A key challenge is to construct the Banach space wisely so that the operator in the abstract problem is densely defined and has the properties necessary to justify the application of the theorems.

The quantity $\alpha$ will then be allowed to vary, thus generalizing the problem. First, $\alpha$ will be allowed to vary with the space variable. This case does not require any additional theory.

The next generalization will be the introduction of a Kelvin-Voigt damping term. In this case the equation has the form

$$u_{tt} + (\beta u_{txx})_{xx} + \alpha u_{xxxx} = 0. \tag{10}$$

The underlying assumption of Kelvin-Voigt damping is that the damping depends on velocity in the same way the basic equations depends on position. A positive value of $\beta$ represents damping in the system. Negative values of $\beta$, which represent energy input

or energy generation, will not be considered. To maintain the step by step approach to generalization, $\alpha$ is held constant while the damping term is considered. Separate treatments will be given to the cases of

1. constant coefficient of damping,
2. spatially varying coefficient of damping,
3. temporally varying coefficient of damping,
4. combined space and time variation of the coefficient of damping, and
5. solution dependent coefficient of damping.

It is in the last of these that a true nonlinearity appears. The form of the equation with this nonlinearity is such that the term quasi-linear is appropriate. Interest in this particular equation motivated the current research.

In another generalization, $\alpha$ will be allowed to vary with time. This case requires a more powerful existence theorem. A more general theorem, which is a synthesis of two theorems of Kato and covers this case, is presented.

### *1.3 A review of the literature*

The review of the literature can be divided into three basic areas. First, a general overview of some problems which use the same general type of theory as that which will be used in this work is given. Second, some problems which are closely related (at least in general appearance) to the current problem are presented. Finally, a review of the literature which provides the theoretical framework for this work is presented.

#### *1.3.1 A sampling of problems from the literature*

Higher order equations are used to describe beam behavior. Ball [3:pg 399] describes a model which accounts for several affects. The equation is

$$\begin{aligned} u_{tt} + \alpha u_{xxxx} - \left(\beta + k\int_0^L u_{\hat{x}}^2 d\hat{x}\right) u_{xx} + \gamma u_{txxxx} & \\ -\sigma \int_0^L u_{\hat{x}} u_{t\hat{x}}\, d\hat{x}\, u_{xx} + \delta u_t &= 0 \end{aligned} \tag{11}$$

He discusses existence, uniqueness, and regularity for the constant coefficient problem. The approach is more classical than modern (*ie* not a semigroup approach).

Pivovarchik [59:pg 647] considers the spectrum of a similar looking equation, namely

$$\alpha u_{txxxx} + u_{xxxx} + (g(x)u_x)_x + k(x)u_t + u_{tt} = 0. \tag{12}$$

More specific examples from the literature will be presented shortly. A brief pause to make some general comments is appropriate.

The current literature contains a wide variety of problems related to fluid flow, plate vibration, beam vibration, and the control of these kinds of phenomena. Since small alterations often make problems which are mathematically distinct, this literature is quite voluminous. No complete review will be attempted. However, it is certainly appropriate to present a sample of the kind of work that is being done.

A brief list of the kinds of items involved in the set up of a problem is now given. A small change in any of these items can lead to entirely different solution behavior. The domain may be bounded or unbounded, fixed or changing, and the conditions specified at the boundary can have a variety of forms. The operators which appear may be bounded or unbounded, linear or nonlinear, autonomous or nonautonomous, they may be self-adjoint, skew-adjoint, compact, or closed. The underlying space may be Hilbert, Banach, normed, linear, metric, or just a set. The order of the equation is significant.

For all of these factors, small changes can affect whether the problem is well-posed, whether it has a solution, whether solutions are unique, what kind of algorithm can be used to obtain the solution, or indeed, what notion of solution is appropriate to consider.

A few samples of equations in the recent literature are given.

Bernis [5:pg 227] establishes existence and uniqueness for the equation

$$(-\Delta)^m u + g(x, u) = f \tag{13}$$

on all of $\Re^N$ with certain limitations on $f$ and $g$. (The symbol $\Delta$ is used for the Laplace operator.) He also establishes existence and uniqueness for the parabolic problem

$$u_t + (-\Delta)^m u + \mid u \mid^{p-1} u = f \tag{14}$$

on $\Re^N \times (0,1)$. Again, certain restrictions apply.

The Schrödinger equation, [68:pg 823],

$$\frac{\partial}{\partial t} u + i\Delta u + |u|^{p-1} u = 0 \tag{15}$$

where $i = (-1)^{\frac{1}{2}}$, also receives attention.

Cannarsa et. al. [7:pg 2] treat a damped wave equation

$$u_{tt}(t,x) = \Delta(u(t,x) + \kappa u_t(t,x)) + f(t,x) \tag{16}$$

on a bounded domain with appropriate boundary and initial conditions. They note (pg 3) that even a slight change in boundary conditions sends the problem into the realm of the unknown. Weak solutions are obtained following a transformation which changes the problem to one where the operator is the generator of an analytic semigroup. They claim this approach can be extended to higher order (*eg* beam) equations.

Kuttler and Hicks [42:pg 1] address existence and uniqueness of a global weak solution to

$$m_{tt} + (P(m_x))_x - (\alpha(m_x) m_{xt})_x = f(t,x). \tag{17}$$

Their emphasis is on time dependent boundary conditions.

Other uses of similar theory are found in Euler Equations [2:pp 367-382], porous medium systems [21:pg 86], and Navier-Stokes equations [36:pg 891]. But, it is time to turn to equations more closely related to the current work.

### *1.3.2 Review of literature related to the current problem*

Fitzgibbon [15:pg 536] addresses (11) as a specific example that comes under an abstract formulation that he gives for a class of quasilinear evolution equations. He examines

$$u_{tt} + \alpha A u_t + Au = F(t, u, u_t) \tag{18}$$

with appropriate initial conditions. The operator $A$ is allowed to be unbounded, but it does not depend on $t$ or $u$. Also, $\alpha$ is a constant. When $A$ is chosen to correspond to the problem of Ball, it turns out that $A$ is self-adjoint and $-A$ is the generator of an analytic semigroup, see also [71:pp 631-633]. Existence and uniqueness results are obtained.

Huang [23:pg 714] discusses the closely related equation

$$u_{tt} + Bu_t + Au = 0 \tag{19}$$

where $B$ is related to $A$ in a certain way.

Authors concerned with control theory still tend to use the more basic forms of the equation while they concentrate on progress in the area of controls. For example, Lasiecka and Triggiani [44:pg 330] and [45:pg 1] use the relatively .mple form

$$u_{tt} + \Delta^2 u = 0. \tag{20}$$

(See also [39:pg 288], for a classical treatment)

Some recent uses of Kelvin-Voigt damping appear in [4:pg 1] and [6:pg 1391]. The first uses it in solving for a material property as a distributed parameter. The second solves for a displacement and stress field in a solid.

Standard treatments of semigroup theory are in [22], [19], [20], and [58].

Several authors have addressed numerical issues related to the implementation of iterative schemes. An early paper discussing the convergence of difference schemes is [55:pg 321]. For a semigroup style numerical analysis book, see as an example, [64]. Another reference on discrete schemes, with emphasis on time dependent operators, is [57]. Another, with emphasis on nonlinearity, is [26].

### *1.3.3 Some methods of analysis*

Two lines of argument, for analysis of the types of equations that have been mentioned, will now be briefly discussed. Many authors have written on these issues, but it is not necessary to give a comprehensive review. The two lines of argument to be mentioned are due principally to Kato and Crandall. A review of the bibliographies of the articles cited will quickly lead to broad coverage of the topic.

The development of the theory of semigroups goes back several decades. It is appropriate to pick up the story with an article by Kato in 1953 [28]. The equation of interest, in his paper, is

$$\frac{d}{dt}u(t) = A(t)u(t) + f(t). \tag{21}$$

This is called an evolution equation because of the time dependence in the definition of the operator $A$. He identifies conditions for the existence of a unique solution.

In the years following this publication, several generalizations were obtained. Most of the extensions were in the direction of weakening one hypothesis or another. Of course, the goal of such work was to widen the range of applicability of the theory.

The method of proof, of the existence of a solution, involves construction of a sequence of operators which are shown to converge to the operator in the original problem. The modified problems corresponding to each of the new operators are solvable. Results which confirm that the function, to which the sequence of solutions converges, is a solution to the original problem are of interest. In this regard the paper of Trotter [70] and a correction to the proof of one of its theorems by Kato [29] are applicable.

By the mid 60's, nonlinear versions were receiving appreciable attention. Of course, in this more complex setting, many distinct variations of a problem can cause it to fail to be linear. Hence, the term *nonlinear* is used with several different meanings in the literature. In some cases it simply means that a function is set-valued rather than single-valued. If the forcing function depends on the solution the term semilinear is used. When the operator depends on the solution, the term quasilinear is used. The terms genuinely nonlinear and fully nonlinear are also encountered. Any of the above terms may be abbreviated to

*nonlinear* in the literature. Hence, it is important to be wary of one's own preconceived notions of what the word suggests.

The 1971 paper of Crandall and Liggett [8] is considered definitive in bringing the nonlinear problem well in hand. It establishes convergence of an exponential type limit which converges to the solution, if one exists.

Also in the early 70's, Kato was publishing in the area of linear evolution equations of hyperbolic type [31] [32]. Recall that if the operator of an equation generates a strongly continuous semigroup, then the equation is called hyperbolic. An equation whose operator is the generator of an analytic semigroup is called parabolic.

By the mid 70's, Kato was working on quasilinear evolution equations of hyperbolic type [34] [35]. In these papers, conditions are given for existence and uniqueness of solutions and several applications are discussed. Specific applications are discussed in [24] and [33].

Through the 80's, Crandall and Souganidis have published [9], [10], and [11] in the area of nonlinear equations. Their approach is to start with a difference scheme as an alternate formulation of the problem. Then the convergence of the scheme is addressed. This approach appears to be more independent of the equation's type. Further, the nonlinear theory is developed directly as opposed to being an extension of the linear theory. Special effort is made to show that situations which satisfy Kato's hypotheses also satisfy those of Crandall. But only in [11] does Crandall claim to have convergence results comparable to those of Kato.

In the present work, emphasis is on the theorems of Kato from [24] and [35]. An excellent text by Pazy [58] incorporates, in well summarized form, a large portion of Kato's work. For convenience, the text of Pazy will be cited for the introduction of terminology and most of the preliminary results. Another text, which includes a summary of Kato's work is [27:pp 237-247].

Other works to broaden perspective include [14] and [53].

### 1.4 *Outline of what is to follow*

The second chapter provides a discussion of the abstract version of (9). The discussion includes basic concepts of the abstract theory and theorems which are sufficient to establish the existence and uniqueness of a solution. Much of the work done in this setting is key to the solution of the more general problem. A linear space and some operators are carefully chosen and shown to have desirable properties. The abstract version of the problem that has been formulated is shown to satisfy the hypotheses of appropriate theorems. Kelvin-Voigt damping is then introduced, leading to the equation

$$u_{tt} + (\beta u_{txx})_{xx} + (\alpha u_{xx})_{xx} = 0 \tag{22}$$

and the individual cases of $\beta$ a constant and $\beta$ dependent on $x$ are considered.

In the third chapter, $\alpha$ will be held constant and time dependent $\beta$ will be considered. This requires additional terminology and theory, which is presented. The cases $\beta$ depends on $t$, $\beta$ depends on $x$ and $t$, and finally, $\beta$ depends on $u$ are also considered.

The fourth chapter considers a case with $\alpha$ dependent on $t$. This requires more theory, which will be presented.

The final chapter summarizes conclusions and recommends specific further work.

Citations to the literature are in brackets and can thus be distinguished from references to equations. Throughout the paper the end of a proof is indicated by a box, like this. $\square$

# II. *Abstract formulations*

In this chapter the basic cases are considered. There is quite a bit of terminology to introduce. This is done in the context of solving the constant coefficient problem. Several basic theorems are also presented.

## *2.1 Preliminaries for the constant coefficient case with no damping*

The differential equation derived in Appendix A to approximate the unforced transverse vibration of a beam with no damping is

$$\tilde{u}_{tt} + D_x^2(\alpha D_x^2 \tilde{u}) = 0 \tag{23}$$

where $D_x = \frac{\partial}{\partial x}$ and $\alpha > 0$ represents $E^r/m$. The case for constant $\alpha$ is not difficult. Indeed, it is easily solved directly by product separation of variables, *eg* [49:pg 117-120]. Nevertheless, this simple case provides an opportunity to introduce terminology, notation, and a strategy for solution which will also be applicable to more complex cases.

The equation will be reformulated as a system of first order equations in $t$. Let $u(t) = \begin{pmatrix} \tilde{u}(t,\cdot) \\ \tilde{u}_t(t,\cdot) \end{pmatrix}$ where the $x$ dependence is suppressed. Explicitly, the values of $u(t)$ are elements of a function space which carries the $x$ dependence. For convenience, the usual notation will be $u = \begin{pmatrix} u_1 \\ u_2 \end{pmatrix}$. The functions $u_1$ and $u_2$, when evaluated at any point in their domain, are required to satisfy appropriate boundary conditions.

The single higher order equation is now replaced with the first order system

$$u_t + Au = 0; \;\; t \geq 0, \; u(0) = u_0 \tag{24}$$

where the vector variable $u$ depends only on $t$. In order to keep the independent variables clear, it is reemphasized that while $u_1$ and $u_2$ are explicit functions of $t$, their values at a particular value of $t$ are functions of $x$. The vector variable $u$ and the boundary conditions

on its components are given by

$$u = \begin{pmatrix} u_1 \\ u_2 \end{pmatrix}; \; u_1(0) = u_1'(0) = u_1(1) = u_1'(1) = 0. \tag{25}$$

The operator $A$ is given by

$$A = A_1 = \begin{pmatrix} 0 & -1 \\ \alpha D^4 & 0 \end{pmatrix}. \tag{26}$$

The subscript on $A$ is used to distinguish this particular operator from others that will be introduced later. When $A$ appears without a subscript, no specific operator is intended. The operator $D$ is $\frac{d}{dx}$. The symbol 0 is used as a vector where appropriate without any special notation. Also, the symbol 1 is used for the identity operator without special notation. The domain of $A_1$ will be denoted by $D(A_1)$. The beam is assumed to be finite, with length 1. The boundary conditions are as indicated.

### *2.1.1 Some definitions*

Basic terminology is now reviewed to establish a foundation for discussion of the abstract problem. The first definition is that of a semigroup [58:pg 1].

**Definition 1** *Let X be a Banach space. A one parameter family S(t),* $0 \le t < \infty$*, of bounded linear operators from X into X is a semigroup of bounded linear operators on X if*

1. *S(0) = I, (the identity operator)*
2. *S(t+s) = S(t)S(s) for every t,s* $\ge 0$.

There are several notions of continuity for semigroups. In this work, strongly continuous semigroups [58:pg 4] are used almost exclusively.

**Definition 2** *A semigroup S(t),* $0 \le t < \infty$*, of bounded linear operators on X is a strongly continuous semigroup (abbreviated* $C_0$ *semigroup). of bounded linear operators if*

$$\lim_{t \downarrow 0} S(t)x = x \text{ for every } x \in X. \tag{27}$$

The terminology refers freely to semigroups of bounded linear operators (even when the generators are not). This is appropriate, as the following theorem shows.

**Theorem 3** *Let $S(t)$ be a $C_0$ semigroup. There exist constants $\omega \geq 0$ and $M \geq 1$ such that*

$$\|S(t)\| \leq Me^{\omega t} \text{ for } 0 \leq t < \infty. \tag{28}$$

**Proof:** See [58:pg 4]. □

If $\omega \leq 0$ the semigroup is said to be uniformly bounded. If, in addition, $M = 1$ then the semigroup is called a $C_0$ semigroup of contractions.

It may be possible to identify a generator (conceptually very much like a derivative) for a given semigroup [58:pg 1].

**Definition 4** *The infinitesimal generator of a semigroup $S(t)$ is the linear operator $A$ defined by*

$$Ax = \lim_{t \downarrow 0} \frac{S(t)x - x}{t} , \tag{29}$$

*whenever this limit exists.*

Notice the conceptual similarity of the generator to a derivative. Not all linear operators are the generators of semigroups. And, some semigroups may have infinitesimal generators which are only defined on a portion of their domain. The notion of an infinitesimal generator (often referred to simply as a generator) is useful when its domain is dense in the domain of definition of the semigroup; *ie*, the space $X$ mentioned in the first two definitions. The collection of all operators $A$ such that $-A$ is the infinitesimal generator of a semigroup on $X$ bounded by a particular $M,\omega$ pair is denoted by $G(X, M, \omega)$.

Later, it will be useful to invert operators of the form $I - \frac{1}{\lambda}A$ for rational $\lambda > 0$. It is appropriate to identify the values of $\lambda$ such that this inverse exists. Special attention is given to values of $\lambda$ for which the inverse is defined on all of X. Some appropriate terms are now introduced [58:pg 8].

**Definition 5** *The resolvent set, $\rho(A)$, of $A$ is the set of all complex numbers $\lambda$ for which $\lambda I$ - $A$ is invertible. That is, $(\lambda I - A)^{-1}$ is a bounded linear operator on $X$.*

**Definition 6** *The family $R(\lambda; A) = (\lambda I - A)^{-1}$, for all $\lambda \in \rho(A)$, of bounded linear operators is called the resolvent of $A$.*

### *2.1.2 Choosing an appropriate space X*

The dependent space variable is suppressed in the notation of (24). In the original formulation $\tilde{u} = \tilde{u}(t, x)$, but now the problem is formulated in a linear space, $X$. The elements of $X$ are functions of $x$ which satisfy the boundary conditions of the problem. Thus $u$ is a function of $t$ and assigns, to each value of $t$, a unique element of $X$. The $x$ dependence is hidden in the domain and is not explicit in the function.

It is reasonable to think of (24) as an ordinary differential equation over a linear space. This is referred to as the abstract formulation of the differential equation and is known as an abstract Cauchy problem. To be specific, choose

$$X = \left\{ y = \begin{pmatrix} y_1 \\ y_2 \end{pmatrix} \in H^2[0,1] \times H^0[0,1] \mid y_1 \text{ satisfies boundary conditions } (25) \right\}. \quad (30)$$

Here $H^p$ represents the standard Sobolev space (see Definition 8). Notice that $X$ is a vector space with each component coming from a Sobolev space. Tl _ boundary conditions are as specified in (25). The following lemma allows an alternate description of $X$.

**Lemma 7** *If $y \in H^2$ and $y(0) = y(1) = y'(0) = y'(1) = 0$, then $y \in H_0^2$.*

**Proof:** See Appendix D. □

The particular boundary conditions under consideration make it appropriate to apply Lemma 7 and describe $X$ in the abbreviated form

$$X = \left\{ y = \begin{pmatrix} y_1 \\ y_2 \end{pmatrix} \in H_0^2[0,1] \times H^0[0,1] \right\}. \quad (31)$$

The domain $[0, 1]$ will often be omitted in the sequel. However, it is intended for the entirety of the paper.

For convenience, the standard Sobolev norm is defined. At this point the Sobolev norm is only used to identify functions which are components of elements in the set $X$. Points in the linear space $X$ will have a norm to be specified momentarily.

**Definition 8** *The standard ($L_2$ style) Sobolev space $H^p$ consists of those functions $u$ such that*

$$\sum_{i=0}^{p} \|D^i u\|_{L_2}^2 < \infty \tag{32}$$

*where $D^0 u = u$ and the norm is given by*

$$\|u\|_{H^p} = \left( \sum_{i=0}^{p} \|D^i u\|_{L_2}^2 \right)^{1/2}. \tag{33}$$

The points (pairs of functions) in $X$ have been identified and it is easy to see that they form a linear space. It is desirable to have a Banach space, and hence, a norm must be specified for the elements of $X$. Alternatively, it is appropriate to specify an inner product for the elements of $X$ and let the norm be the one naturally induced by it. An inner product will be specified and then it will be shown that $X$ is a complete space under the induced norm. A preliminary lemma is required.

**Lemma 9** *If $z$ is absolutely continuous on [0,1] and $z'(x) = 0$ almost everywhere, then $z$ is constant.*

**Proof:** See [65:pg 105]. □

An inner product is now presented. This inner product has been used previously in [12:pg 16] and [35:pp 144, 147].

**Theorem 10** *The expression*

$$\left( \begin{pmatrix} u_1 \\ u_2 \end{pmatrix}, \begin{pmatrix} v_1 \\ v_2 \end{pmatrix} \right) = \int_0^1 \alpha D^2 u_1 D^2 v_1 \, dx + \int_0^1 u_2 v_2 \, dx \tag{34}$$

*where $\alpha$ is a positive constant, defines an inner product on $X$.*

**Proof:** This is shown to be an inner product in a straightforward fashion from the definition (to review the definition, see, for example [54:pg 272], [65:pg 210], or [73:pp 39-40]). The necessity of having the boundary conditions in the definition of $X$ becomes clear in the details.

The only issue in doubt is that of positive definiteness. Suppose that

$$\begin{aligned} \left( \begin{pmatrix} u_1 \\ u_2 \end{pmatrix}, \begin{pmatrix} u_1 \\ u_2 \end{pmatrix} \right) &= \int_0^1 \alpha D^2 u_1 D^2 u_1 \, dx + \int_0^1 u_2 u_2 \, dx \\ &= \alpha \int_0^1 (D^2 u_1)^2 \, dx + \int_0^1 u_2^2 \, dx \\ &= 0. \end{aligned} \tag{35}$$

This requires each integral to be zero, and hence $u_1''$ and $u_2$ to be zero, at least almost everywhere (a.e.). Since functions which only differ on a set of measure zero are treated as identical, it is clear that $u_2$ is zero. If the boundary conditions force $u_1 = 0$, then the proposed inner product is legitimate.

Lemma 9 is now applied to show that if $z \in H^2$ and $z'' = 0$ a.e., then $z = 0$. Indeed, for $z$ to be an element of $H^2$ requires $\int_0^1 (z'')^2 \, dx$ to exist. Since the domain of integration is bounded, $\int_0^1 z'' \, dx$ exists. It follows that $z'$ is absolutely continuous (a.c.). Since $z'$ is a.c. and $z'' = 0$ a.e., the lemma says $z'$ is constant. The boundary conditions on $z'$ force the constant to be zero. Thus, $z$ is constant. The boundary conditions on $z$ force this constant to be zero also. Apply the lemma to $u_1$ and positive definiteness is clear.

This completes the proof of the theorem. □

The norm chosen for $X$ is $\|\cdot\|_X = (\cdot,\cdot)^{1/2}$. The set $X$ with this norm is the Banach space that will be referred to in the sequel. (Since the norm comes from an inner product it would be appropriate to call [illegible] Hilbert space. However, the more general term Banach space will usually be used.)

### *2.1.3 Completeness of X*

It is important that $X$ be complete. This issue will now be addressed. Several preliminaries are needed. As a matter of notation, $\|\cdot\|_\infty$ denotes the supremum norm.

**Lemma 11** *If $z \in H^2$ then $\|z\|_{L_2} \le \|z\|_\infty$.*

**Proof:** It is important for the proof that the function have a bounded domain of definition.

$$\begin{aligned} \|z\|_{L_2} &= \left(\int_0^1 z^2(x)\,dx\right)^{1/2} \\ &\le \left(\int_0^1 \|z\|_\infty^2\,dx\right)^{1/2} \\ &= \|z\|_\infty. \quad \square \end{aligned} \tag{36}$$

**Lemma 12** *If $z \in H^2$ then, $\|z\|_\infty \le |z(0)| + \|z'\|_{L_2}$.*

**Proof:** Since $z \in H^2$, $z'$ is square integrable. Since the domain is a bounded interval, it follows that $z'$ is integrable. Then [65:pg 101] $z$ is given (a.e.) by

$$z(x) = z(0) + \int_0^x z'(\hat{x})d\hat{x}. \tag{37}$$

It follows that

$$\begin{aligned} |\, z(x) \,| &= \left| z(0) + \int_0^x z'(\hat{x})d\hat{x} \right| \\ &\le |\, z(0) \,| + \left| \int_0^x z'(\hat{x})d\hat{x} \right| \\ &\le |\, z(0) \,| + \int_0^x |\, z'(\hat{x}) \,|\, d\hat{x} \\ &\le |\, z(0) \,| + \int_0^1 |\, z'(\hat{x}) \,|\, d\hat{x} \\ &\le |\, z(0) \,| + \|z'\|_{L_2}. \end{aligned} \tag{38}$$

The Schwarz inequality has been used in the last step. Since the right hand side is free of $x$ it follows that

$$\|z\|_\infty \le |\, z(0) \,| + \|z'\|_{L_2} \tag{39}$$

as desired. $\square$

This lemma has an obvious corollary.

**Corollary 13** *If $z \in H^2$ and $z(0) = 0$, then $\|z\|_\infty \le \|z'\|_{L_2}$.*

A similar lemma and corollary, for $z'$ instead of $z$, follow immediately. When $z(0) = z'(0) = 0$, these combine to give

$$\|z\|_{L_2} \leq \|z\|_\infty \leq \|z'\|_{L_2} \leq \|z'\|_\infty \leq \|z''\|_{L_2}. \tag{40}$$

When the above lemmas apply, the following composite result is available.

**Lemma 14** *For the given norm on $X$, $\|z\|_{H_0^2} \leq \left(\frac{3}{\alpha}\right)^{1/2} \left\| \begin{pmatrix} z \\ 0 \end{pmatrix} \right\|_X$.*

**Proof:** This is proved by using the previous results.

$$\begin{aligned}
\|z\|_{H_0^2} &= \left(\|z\|_{L_2}^2 + \|z'\|_{L_2}^2 + \|z''\|_{L_2}^2\right)^{1/2} \\
&\leq \left(3\|z''\|_{L_2}^2\right)^{1/2} \\
&= \left(\frac{3}{\alpha} \left\| \begin{pmatrix} z \\ 0 \end{pmatrix} \right\|_X^2 \right)^{1/2} \\
&= \left(\frac{3}{\alpha}\right)^{1/2} \left\| \begin{pmatrix} z \\ 0 \end{pmatrix} \right\|_X. \quad \square
\end{aligned} \tag{41}$$

This concludes the preliminaries for an argument on the completeness of $X$.

**Theorem 15** *The space $X$, with the inner product introduced in Theorem 10, is complete.*

**Proof:** Let $\{y_n\}$ be a Cauchy sequence in $X$. Recall that elements of $X$ have two components. To be explicit,

$$\{y_n\} = \left\{ \begin{pmatrix} y_{1_n} \\ y_{2_n} \end{pmatrix} \right\}. \tag{42}$$

This will converge to a point $y = \begin{pmatrix} y_1 \\ y_2 \end{pmatrix} \in X$ if and only if the first components of the given sequence converge to $y_1 \in H_0^2$, and the second components converge to $y_2 \in H^0$. Convergence will be addressed for each component separately. Keys to the strategy are

the relationship

$$\begin{pmatrix} y_{1_n} \\ y_{2_n} \end{pmatrix} = \begin{pmatrix} y_{1_n} \\ 0 \end{pmatrix} + \begin{pmatrix} 0 \\ y_{2_n} \end{pmatrix} \tag{43}$$

and the fact that (for the specified norm on $X$) that

$$\left\| \begin{pmatrix} y_{1_n} \\ y_{2_n} \end{pmatrix} \right\|_X = \left\| \begin{pmatrix} y_{1_n} \\ 0 \end{pmatrix} \right\|_X + \left\| \begin{pmatrix} 0 \\ y_{2_n} \end{pmatrix} \right\|_X . \tag{44}$$

Since the left hand side forms a Cauchy sequence, it is clear that each term on the right hand side must form a Cauchy sequence. Now focus attention on the second term.

For each $n$, $y_{2_n} \in H^0$. Furthermore,

$$\left\| \begin{pmatrix} 0 \\ y_{2_n} \end{pmatrix} \right\|_X = \|y_{2_n}\|_{H^0}. \tag{45}$$

Hence, $\{y_{2_n}\}$ is Cauchy in $\|\cdot\|_{H^0}$. But, $H^0$ is complete by definition, *eg* [1:pg 44]. Hence, there exists some $y_2 \in H^0$, $y_{2_n} \to y_2$. This is the desired $y_2$.

The case of $y_1$ is nearly as simple. Each element of $\{y_{1_n}\}$ is in $H_0^2$, and the strategy is to find $y_1$ in the complete space $H_0^2$ such that $y_{1_n} \to y_1$. But, this depends on $\{y_{1_n}\}$ being a Cauchy sequence in the norm of $H_0^2$. This follows immediately from Lemma 14. Thus $\{y_{1_n}\}$ is Cauchy as a sequence in the complete space $H_0^2$ and converges, say to $y_1$.

It is easy to see that $\begin{pmatrix} y_1 \\ y_2 \end{pmatrix} \in X$ and it follows that $X$ is complete. This completes the proof of the theorem. □

It is worth noting that the boundary conditions were important in these arguments. Extension of the result to other boundary conditions is not trivial. Additional comments on this point are in Appendix B.

#### *2.1.4 Identifying the domain of $A_1$*

It is important to identify the portion of $X$ on which $A_1$ makes sense and has its image again in $X$. Let

$$D(A_1) = \left\{ y = \begin{pmatrix} y_1 \\ y_2 \end{pmatrix} \in H^4 \times H_0^2 \mid y_1 \text{ satisfies boundary conditions (25)} \right\}. \tag{46}$$

Notice that the second components satisfy the boundary conditions. This is expressed in the $H_0^2$ notation and omitted from the specifier. Recall that the boundary conditions were specified in (25).

Several comments are appropriate to prepare to establish that the domain of $A_1$ is dense in $X$. First, note that it is sufficient to consider the components individually. Second, in the way of notation, the collection of all functions which are infinitely differentiable and have compact support on $(0,1)$ will be denoted $C_0^\infty$. (It would be more standard to write $C_0^\infty(0,1)$ so there is some abbreviation here.) Third, the collection of functions in $H^p$ which satisfy the boundary conditions will be denoted $H_\partial^p$.

Theorem 16 *For $A_1$ in (26), $D(A_1)$ is dense in $X$.*

**Proof:** The skeleton of an appropriate argument has been presented in [12:pg 19]. For denseness of the first component it is sufficient to establish that $\overline{H_\partial^4} \supset H_\partial^2$. Note that

$$C_0^\infty \subset H_\partial^4 \subset H_\partial^2 \subset H_0^2 = \overline{C_0^\infty} \tag{47}$$

where the closure is with respect to the $H^2$ norm. The first two inclusions are clear. The last inclusion follows from Lemma 7. The equality holds by definition, *cg* [1:pg 45].

Note that $\overline{H_\partial^4} = H_0^2$ and that $H_\partial^4$ is the first component of $D(A_1)$. Also, $H_0^2 \supset H_\partial^2$ and hence $\overline{H_\partial^4} \supset H_\partial^2$. Denseness of the first components is now clear.

For the second component it is sufficient to establish that $\overline{H_\partial^2} \supset H^0$. Consider

$$C_0^\infty \subset H_\partial^2 \subset H^2 \subset H^0 = L_2 \subset \overline{C_0^\infty} \tag{48}$$

where the closure is with respect to the $L_2$ norm. The last containment follows from Theorem 2.19 of [1:pg 31] (see also [20:pg 253] or [54:pg 592]). Now the denseness is clear.

Since each component of $D(A_1)$ is dense in the corresponding component of $X$, it is clear that $D(A_1)$ is dense in $X$. □

**Corollary 17** *The inclusions $H_\partial^4 \subset H_0^2 \subset H^0$ are dense.*

### *2.1.5 Special properties of $A_1$ and its adjoint*

Later, there will be a need for the adjoint, $A_1^*$, of $A_1$. Initial definitions and computations are presented now. See [54:pp 352, 527], [63:pp 201, 215], or [73:pg 196] for details.

**Definition 18** *The operator $A^*$ is called the adjoint of the operator $A$ if $(Au, v) = (u, A^*v)$ for all $u, v \in D(A)$.*

The following derivation identifies the operator $A_1^*$ for $A_1$ defined by (26).

$$\begin{aligned}
\left(A_1\begin{pmatrix} u_1 \\ u_2 \end{pmatrix}, \begin{pmatrix} v_1 \\ v_2 \end{pmatrix}\right) &= \left(\begin{pmatrix} -u_2 \\ \alpha D^4 u_1 \end{pmatrix}, \begin{pmatrix} v_1 \\ v_2 \end{pmatrix}\right) \\
&= -\alpha \int_0^1 D^2 u_2 D^2 v_1 \, dx + \alpha \int_0^1 v_2 D^4 u_1 \, dx \\
&= -\alpha D u_2 D^2 v_1 \,|_0^1 + \alpha \int_0^1 D u_2 D^3 v_1 \, dx \\
&\quad + \alpha v_2 D^3 u_1 \,|_0^1 - \alpha \int_0^1 D v_2 D^3 u_1 \, dx \\
&= \alpha u_2 D^3 v_1 \,|_0^1 - \alpha \int_0^1 u_2 D^4 v_1 \, dx \\
&\quad - \alpha D v_2 D^2 u_1 \,|_0^1 + \alpha \int_0^1 D^2 v_2 D^2 u_1 \, dx \\
&= \left(\begin{pmatrix} u_1 \\ u_2 \end{pmatrix}, \begin{pmatrix} v_2 \\ -\alpha D^4 v_1 \end{pmatrix}\right) \\
&= \left(\begin{pmatrix} u_1 \\ u_2 \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ -\alpha D^4 & 0 \end{pmatrix}\begin{pmatrix} v_1 \\ v_2 \end{pmatrix}\right) \qquad (49)
\end{aligned}$$

where all of the boundary terms are zero and are dropped after their first appearance.

It is interesting that $A_1^* = -A_1$. When this occurs, the operator $A_1$ is called skew-adjoint. Since $D(A_1) = D(-A_1)$ it follows that $D(A_1^*) = D(A_1)$. Thus, since $A_1$ is densely defined, so is $A_1^*$. (Actually, it is not quite this straightforward. Special care is required in describing the domain of an unbounded operator. For $A_1$ it is more complete to argue as follows: Let $y \in D(A_1)$ be given. The goal is to show that $y \in D(A_1^*)$. This requires the existence of some $z \in X$ such that for every $x \in D(A_1), (A_1x, y) = (x, z)$. Let $x \in D(A_1)$ be given. By direct computation $(A_1x, y) = (x, -A_1y)$. Choose $z = -A_1y$. Since $y \in D(A_1)$ and $D(A_1) = D(-A_1)$ it follows that $z \in X$. Thus $y \in D(A_1^*)$ and $D(A_1) \subset D(A_1^*)$. The argument here is reversible and the containment goes both ways. The equality of the two domains is now clear.) The issue of $A_1^*$ actually being defined on the dual space has been ignored since $X$ is a Hilbert space and is identified with its dual (*eg* [73:pg 91]).

Another useful concept is that of the dissipative operator [58:pg 14].

**Definition 19** *A linear operator $A$ is dissipative if $\|(\lambda I - A)u\| \geq \lambda\|u\|$ for all $\lambda > 0$ and all $u \in D(A)$.*

The idea of an operator being dissipative is quite simple. If $Au + u$ is in some way less than $u$, then it is quite reasonable to call $A$ dissipative. If $u$ were a vector it would be reasonable to think of $Au$ as having a component in the direction of $-u$. This could be validated algebraically by considering the inner product and requiring $(Au, u) \leq 0$. For a definition that is acceptable in a Banach space it is reasonable to require $u - Au$ to be greater, in some sense, than $u$. If, for every $\lambda > 0$, it happens that $\lambda u - Au$ is bigger than $\lambda u$, then it is certainly reasonable to call $A$ dissipative. This is just what the definition does.

The following lemma shows that, in a Hilbert space, the desired implication holds.

**Lemma 20** *If $(Au, u) \leq 0$ for every $u \in D(A)$, then $A$ is dissipative.*

**Proof:** Basic definitions are sufficient to establish the implication.

$$\|(\lambda I - A)u\| = ((\lambda I - A)u, (\lambda I - A)u)^{1/2}$$

$$
\begin{aligned}
&= (\lambda Iu - Au, \lambda Iu - Au)^{1/2} \\
&= ((\lambda u, \lambda u) - (\lambda u, Au) - (Au, \lambda u) + (Au, Au))^{1/2} \\
&= (\lambda^2(u,u) - 2\lambda(Au,u) + (Au,Au))^{1/2} \\
&\geq (\lambda^2(u,u) - 2\lambda(Au,u))^{1/2} \\
&\geq (\lambda^2(u,u))^{1/2} \\
&= \lambda\|u\|. \qquad \square
\end{aligned} \tag{50}
$$

It is useful to establish that the operators $A_1$ and $A_1^*$ are dissipative. First, consider $A_1$.

**Lemma 21** *The operator $A_1$ in (26) is dissipative.*

**Proof:** Proof is by direct computation.

$$
\begin{aligned}
\left(A_1 \begin{pmatrix} u_1 \\ u_2 \end{pmatrix}, \begin{pmatrix} u_1 \\ u_2 \end{pmatrix}\right) &= -\alpha \int_0^1 D^2u_2 D^2u_1 \, dx + \alpha \int_0^1 u_2 D^4 u_1 \, dx \\
&= -\alpha D u_2 D^2 u_1 \mid_0^1 + \alpha \int_0^1 D u_2 D^3 u_1 \, dx \\
&\quad + \alpha u_2 D^3 u_1 \mid_0^1 - \alpha \int_0^1 D u_2 D^3 u_1 \, dx \\
&= 0.
\end{aligned} \tag{51}
$$

where the boundary terms are again zero. This is sufficient, by Lemma 20, to establish that $A$ is dissipative. $\square$

It is immediately clear that $-A_1$, $A_1^*$, and $-A_1^* = (-A_1)^*$ are all dissipative. This is a convenience of having an equality when an inequality suffices in (51). Of course this was not accidental, the choice of inner product made it happen. It is interesting to note how such an inner product is chosen.

If the usual $L_2$ inner product is used, then the following computations are sufficient to check whether $A_1$ is dissipative.

$$\begin{aligned}
\left(A_1\begin{pmatrix} u_1 \\ u_2 \end{pmatrix}, \begin{pmatrix} u_1 \\ u_2 \end{pmatrix}\right) &= \left(\begin{pmatrix} 0 & -1 \\ \alpha D^4 & 0 \end{pmatrix}\begin{pmatrix} u_1 \\ u_2 \end{pmatrix}, \begin{pmatrix} u_1 \\ u_2 \end{pmatrix}\right) \\
&= \left(\begin{pmatrix} -u_2 \\ \alpha D^4 u_1 \end{pmatrix}, \begin{pmatrix} u_1 \\ u_2 \end{pmatrix}\right) \\
&= -\int_0^1 u_2 u_1\, dx + \int_0^1 \alpha u_2 D^4 u_1\, dx \qquad (52)
\end{aligned}$$

This quantity must be less than or equal to 0.

One strategy is to redefine the inner product so that everything cancels. Possible replacements for $-\int_0^1 u_2 u_1\, dx$ would be

$$\begin{aligned}
&-\int_0^1 \alpha u_2 D^4 u_1\, dx, \\
&\int_0^1 \alpha D u_2 D^3 u_1\, dx, \text{ or} \\
&-\int_0^1 \alpha D^2 u_2 D^2 u_1\, dx. \qquad (53)
\end{aligned}$$

Each of these, after the appropriate application of integration by parts would show $A_1$ to be dissipative by making the inner product exactly zero. The last candidate in (53) corresponds to taking $\alpha$ times the $L_2$ inner product of the second derivatives of the first components. It has been verified in Theorem 10 that such a replacement leads to a legitimate inner product.

#### *2.1.6 Closedness of $A_1$*

It is also important to establish that the operators $A_1$ and $A_1^*$ are closed. A definition is first given. See [54:pp 241, 529], [63:pg 300], or [73:pg 77].

**Definition 22** *A linear operator $A$ is closed if, when any pair of sequences $\{x_n\} \subset D(A) \subset X$ and $\{y_n\} = \{Ax_n\} \subset X$ both converge, say $x_n \to x \in X$ and $y_n \to y \in X$ then $x \in D(A)$ and $Ax = y$.*

**Lemma 23** *If $A$ is a closed operator, then so is $-A$.*

**Proof:** This is clear from the linearity of the space and the equality $D(A) = D(-A)$. $\square$

The proof of the theorem that will be used to establish closedness of $A_1$ requires a lemma.

**Lemma 24** *Let $(\cdot,\cdot)$ denote an inner product on a linear space $X$. For fixed $x \in X$, let $f(y) = (x, y)$. Then $f$ is continuous. That is, the inner product is a continuous function of its arguments.*

**Proof:** See [20:pp 179-180] or Appendix D. $\square$

There will be several occasions to use the next result [54:pg 529]. It is actually a special case of the theorem cited. An additional hypothesis that $A$ be closable and an additional result that $D(A^*)$ is dense are omitted. (The proof of the omitted portion is relegated to guided exercises in [54:pp 531-532]. A different proof is in [25:pg 172]. See also [63:pp 299-300].)

**Theorem 25** *Let $A$ be a densely defined linear operator on a Hilbert space $H$. Then the adjoint, $A^*$, of $A$ is a closed operator.*

**Proof:** Let $u_n^* \to u^*$ and $A^*u_n^* \to w^*$ with $u_n^* \in D(A^*)$ for all $n$. By the definition of adjoint, $(Au, u_n^*) = (u, A^*u_n^*)$ for all $u \in D(A)$. From the continuity of the inner product it follows that $(u, w^*) = (Au, u^*)$. But, again recalling the definition of adjoint, $u^* \in D(A^*)$ if there is some $z \in X$ such that $(Au, u^*) = (u, z)$. When this occurs $z = A^*u^*$. Clearly, $z = w^*$ is the element needed and $A^*u^* = w^*$. Thus $u^* \in D(A^*)$.

This establishes that $A^*$ is closed. $\square$

**Lemma 26** *The operator $-A_1$, in (26), is closed.*

**Proof:** Recall that $A_1^* = -A_1$ and that $D(A_1^*) = D(A_1)$. Thus, since $D(A_1)$ is dense in X so is $D(A_1^*)$. (A more general approach would be to apply Theorem 1.4.5.c of [58:pg 15] and Theorem 7.10.3 of [54:pg 529], but this is not necessary in the present case.) Then,

by Theorem 25, $A_1^{**}$ is a closed operator. But $A_1^{**} = A_1$ (It is almost trivial to verify this directly. It is not surprising in the Hilbert space setting, for example see [54:pg 353] for the case of bounded $A$.) so $A_1$ is a closed operator. It is immediate that $-A_1$ is also closed. □

### *2.1.7 The generator of a $C_0$ semigroup of contractions*

It is important that the operator $-A_1$ be the generator of a $C_0$ semigroup of contractions. This is an immediate result of the next theorem.

It is useful to note that the literature is not uniform in the use of $A$ and $-A$. This results from the form in which the original differential equation is written. That is to say $u_t + Au = 0$ and $u_t = Au$ correspond to opposite sign conventions. Also, some authors use accretiveness instead of dissipativeness which leads to opposite sign conventions. It is important to be self-consistent. Nevertheless, it frequently requires a conscious effort to keep this, somewhat annoying detail, straight.

The next theorem provides a useful tool for establishing that certain operators are the infinitesimal generators of $C_0$ semigroups.

**Theorem 27** *Let $A$ be a densely defined closed linear operator. If both $A$ and $A^*$ are dissipative, then $A$ is the infinitesimal generator of a $C_0$ semigroup of contractions on $X$.*

**Proof:** See [58:pg 15] or [40:pg 87, Theorem 4.4]. □

**Corollary 28** *The operator $-A_1$ in (26) is the generator of a $C_0$ semigroup of contractions.*

**Proof:** It has been established in the preceeding subsections that the hypotheses of the theorem are satisfied. Hence $-A_1$ is the generator of a $C_0$ semigroup of contractions. □

## *2.2 Remarks*

It is interesting to note that $A_1$ is, in fact, the generator of a group. (Use Theorem 27 on $A^*$ and $A^{**}$. Also, see [19:pg 22, 2.16, pg 32 Theorem 4.7], and [58:pg 22, Theorem 1.6.5 and pg 41, Theorem 1.10.8].) This fact is not of any immediate interest. However,

it is appropriate to note that this feature can be useful. For an example, see [43:pg 745] where it is used in a controls problem.

A distinctive feature of this work is the attention to hyperbolic problems. This means that $-A_1$ is the generator of a $C_0$ semigroup but not necessarily of an analytic semigroup [35:pp 128-129]. If it were the generator of an analytic semigroup then the theory of parabolic equations would apply, *eg* [16:pg 108]. It is appropriate to verify that the current problem does not fit into the more specialized category. This is proved in Appendix C to avoid too much of a distraction at this point.

## *2.3 Finishing up for constant $\alpha$*

It is time to complete the constant coefficient problem. The preliminaries have been rather complete and have included verification of the hypotheses of the existence theorem to be presented now. The next theorem guarantees the existence of a unique $C^1$ solution to (24) for any $u_0 \in D(A)$. A solution is an $X$ valued function $u(t)$ that is continuous on $[0,\infty)$, continuously differentiable on $(0,\infty)$, has $u(t) \in D(A)$ for all $t > 0$, and satisfies the differential equation (24) for all $t > 0$ [58:pp 105, 139].

### *2.3.1 The existence theorem*

**Theorem 29** *Let the operator $-A$ be the infinitesimal generator of a $C_0$ semigroup $S(t)$. Then (24) has a unique solution $u(t)$, which is continuously differentiable on $[0,\infty)$, for every initial value in $D(-A)$.*

**Proof:** See [58:pp 102-104]. □

**Theorem 30** *The abstract Cauchy problem (24), with $A$ replaced by $A_1$, has a unique classical solution.*

**Proof:** To see that the hypotheses of the preceeding theorem are satisfied, it suffices to review Theorem 16 and Theorem 27 (which in turn depends on Lemma 21 and the comments following it). This theorem now follows from Theorem 29. □

This completes the existence and uniqueness argument for constant $\alpha > 0$. The solution is given by

$$u(t) = S(t)u_0 \tag{54}$$

where $S(t)$ is the semigroup generated by $-A_1$.

### 2.3.2 Continuous dependence

It may be that $u_0$ is not known exactly, or even if it is known, it may be that it cannot be represented exactly. In either of these cases it becomes important to know whether small changes in $u_0$ lead to only small changes in $u(t)$. This is usually referred to as continuous dependence of the solution on the data.

For the current problem, $S(t)$ is a $C_0$ semigroup of contractions. This means that $\|S\| \leq 1$. Hence, the following theorem is straightforward.

**Theorem 31** *The unique classical solution of (24), with A replaced by $A_1$, depends continuously on the initial data.*

**Proof:** Suppose $\hat{u}_0$ is an initial condition, possibly different from $u_0$. Let $\epsilon > 0$ be given. For $t \geq 0$, let $\hat{u}(t) = S(t)\hat{u}_0$. Then

$$\begin{aligned} \|u(t) - \hat{u}(t)\|_X &= \|S(t)u_0 - S(t)\hat{u}_0\|_X \\ &= \|S(t)(u_0 - \hat{u}_0)\|_X \\ &\leq \|S\|_X \, \|u_0 - \hat{u}_0\|_X \\ &\leq \|u_0 - \hat{u}_0\|_X \,. \end{aligned} \tag{55}$$

Choose $\delta = \epsilon$ and the continuity of the dependence is established. □

### 2.3.3 Constructing a solution

Before consideration is given to generalizations of the problem, it is appropriate to consider how the solution, whose existence has just been established, can actually be obtained.

If the operator $-A$ is bounded, and the infinitesimal generator of a $C_0$ semigroup, then the solution to (24) is given by $u(t) = e^{-tA}u_0$. Since $A_1$ in (26) need not be bounded, limit definitions (with the same flavor as that of $e^{-tA}$) must be carefully analyzed to assure convergence. But, the conditions which assure that $-A_1$ is the generator of a $C_0$ semigroup do just that. When it exists, $e^{-tA}$ is the semigroup $S(t)$. This motivates the terminology for $-A$ as the *generator* of $S(t)$. The solution always has the form $u(t) = S(t)u_0$. The key issue in applications is to determine when $S(t)$ can be determined from $A$. When $S(t)$ can be obtained, the first component of $u(t)$ is the solution to the original problem.

Here is one way of obtaining a semigroup [58:pg 33].

**Theorem 32** *Let $S(t)$ be a $C_0$ semigroup on $X$. If $A$ is the infinitesimal generator of $S(t)$ then*

$$S(t)x = \lim_{n\to\infty}\left(I - \frac{t}{n}A\right)^{-n} x = \lim_{n\to\infty}\left[\frac{n}{t}R\left(\frac{n}{t} : A\right)\right]^{n} x \tag{56}$$

*for $x \in X$, and the limit is uniform in $t$ on any bounded interval.*

**Proof:** See [58:pp 34-35]. A more general development is presented later in the same reference, [58:pp 89-92]. □

In the event that the operator $A$ is either not known precisely or cannot be represented precisely, it is of interest to know whether the semigroup generated by an approximate $A$ is close to the semigroup that would have been generated by an exact $A$. This has the same flavor as the continuous dependence considered above. Indeed, it would be appropriate to ask whether the semigroup depends continuously on the generator.

Results of this type are obtained by considering a sequence $A_n$ which is assumed to converge to $A$ in an appropriate sense. The question is whether the corresponding semigroups $S_n(t)$ converge to the semigroup generated by $A$. Formal results in this direction often consider also whether $A$ depends continuously on $S$. Results of this type are referred to as Trotter or Trotter-Kato theorems. (Trotter presented the pioneering work in the linear case. Kato corrected an error in the published proof.)

For results applicable to the current problem, see [58:pp 35, 84-89], [19:pp 44, 48], or [73:pg 269]. The results are that three distinct convergences are in fact equivalent: the operators, the semigroups, and the resolvents of the operators.

The works of Crandall, *eg* [10] and [11], provide an alternate approach. The alternate approach will not be pursued here. However, it is appropriate to note that it started with a fundamental paper in 1971 [8]. The key theorem has a somewhat sketchy proof. A detailed version is provided as Appendix E.

### *2.4 Spatially dependent coefficient*

Consider $u_{tt} + D^2(\alpha(x)D^2u) = 0$. Assume that $\alpha$ is continuously differentiable with respect to $x$ and for some $\alpha_{min} > 0$, $\alpha(x) \geq \alpha_{min} > 0$ for all $x \in [0,1]$. Choose the space $X$ and its inner product as before. For the validation of the inner product to go as before, the strict inequality in the requirement $\alpha(x) > 0$ is necessary. This is because (35) must hold with $\alpha$ inside the integral, otherwise the argument is unchanged. This time the operator is defined as follows:

$$A_2 = \begin{pmatrix} 0 & -1 \\ D^2(\alpha(\cdot)D^2) & 0 \end{pmatrix}. \tag{57}$$

Clearly $A_2$ is a linear operator. Note that $D(A_2) = D(A_1)$. In order to determine $A_2^*$ let $u \in D(A_2)$ be given, then

$$\begin{aligned}
\left( A_2 \begin{pmatrix} u_1 \\ u_2 \end{pmatrix}, \begin{pmatrix} v_1 \\ v_2 \end{pmatrix} \right) &= \left( \begin{pmatrix} -u_2 \\ D^2(\alpha(\cdot)D^2)u_1 \end{pmatrix}, \begin{pmatrix} v_1 \\ v_2 \end{pmatrix} \right) \\
&= -\int_0^1 \alpha(x) D^2u_2 D^2v_1\,dx + \int_0^1 v_2 D^2(\alpha(x)D^2u_1)\,dx \\
&= -\alpha(x)Du_2D^2v_1 \,\Big|_0^1 + \int_0^1 Du_2 D(\alpha(x)D^2v_1)\,dx \\
&\quad + v_2D(\alpha(x)D^2u_1)\,\Big|_0^1 - \int_0^1 Dv_2 D(\alpha(x)D^2u_1)\,dx \\
&= u_2D(\alpha(x)D^2v_1)\,\Big|_0^1 - \int_0^1 u_2 D^2(\alpha(x)D^2v_1)\,dx \\
&\quad - Dv_2\alpha(x)D^2u_1\,\Big|_0^1 + \int_0^1 \alpha(x)D^2u_1D^2v_2\,dx
\end{aligned}$$

$$= \left( \begin{pmatrix} u_1 \\ u_2 \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ -D^2(\alpha(x)D^2) & 0 \end{pmatrix} \begin{pmatrix} v_1 \\ v_2 \end{pmatrix} \right) \tag{58}$$

where a$\ddot{\ }$ of the boundary terms are zero.

Observe that $A_2^* = \begin{pmatrix} 0 & 1 \\ -D^2(\alpha(\cdot)D^2) & 0 \end{pmatrix}$ and hence, as before $A_2^* = -A_2$. Since the steps are reversible, it is easy to see that $(A_2^*)^* = -A_2^* = (-A_2)^* = A_2$. By Theorem 25, $-A_2$ is closed.

Next, it is necessary to show that $-A_2$ is dissipative. For $u \in D(A_2)$

$$\begin{aligned}
\left(-A_2 \begin{pmatrix} u_1 \\ u_2 \end{pmatrix}, \begin{pmatrix} u_1 \\ u_2 \end{pmatrix}\right) &= \left( \begin{pmatrix} u_2 \\ -D^2(\alpha(\cdot)D^2 u_1) \end{pmatrix}, \begin{pmatrix} u_1 \\ u_2 \end{pmatrix} \right) \\
&= \int_0^1 \alpha(x) D^2 u_2 D^2 u_1 \, dx - \int_0^1 u_2 D^2(\alpha(x) D^2 u_1) \, dx \\
&= \alpha(x) D u_2 D^2 u_1 \,|_0^1 - \int_0^1 D u_2 D(\alpha(x) D^2 u_1) \, dx \\
&\quad - u_2 D(\alpha(x) D^2 u_1) \,|_0^1 + \int_0^1 D u_2 D(\alpha(x) D^2 u_1) \, dx \\
&= 0
\end{aligned} \tag{59}$$

where again the boundary terms are each zero.

Clearly $A_2$ is also dissipative in this case. But $A_2 = (-A_2)^*$ and thus $(-A)^*$ is dissipative. Now all of the theorems apply as in the case of constant $\alpha$ and guarantee the existence of a unique solution to the differential equation.

### *2.5 Constant coefficient Kelvin-Voigt damping*

Consider $u_{tt} + \beta u_{txxxx} + \alpha u_{xxxx} = 0$ with $u(0,x) = u_0$, $u_t(0,x) = u_{t0}$ with the boundary conditions as before. Throughout this section $\alpha$ is a positive constant. To begin,

$$= \left( \begin{pmatrix} u_1 \\ u_2 \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ -D^2(\alpha(x)D^2) & 0 \end{pmatrix} \begin{pmatrix} v_1 \\ v_2 \end{pmatrix} \right) \tag{58}$$

where all of the boundary terms are zero.

Observe that $A_2^* = \begin{pmatrix} 0 & 1 \\ -D^2(\alpha(\cdot)D^2) & 0 \end{pmatrix}$ and hence, as before $A_2^* = -A_2$. Since the steps are reversible, it is easy to see that $(A_2^*)^* = -A_2^* = (-A_2)^* = A_2$. By Theorem 25, $-A_2$ is closed.

Next, it is necessary to show that $-A_2$ is dissipative. For $u \in D(A_2)$

$$\begin{aligned}
\left( -A_2 \begin{pmatrix} u_1 \\ u_2 \end{pmatrix}, \begin{pmatrix} u_1 \\ u_2 \end{pmatrix} \right) &= \left( \begin{pmatrix} u_2 \\ -D^2(\alpha(\cdot)D^2u_1) \end{pmatrix}, \begin{pmatrix} u_1 \\ u_2 \end{pmatrix} \right) \\
&= \int_0^1 \alpha(x)D^2u_2D^2u_1\,dx - \int_0^1 u_2D^2(\alpha(x)D^2u_1)\,dx \\
&= \alpha(x)Du_2D^2u_1 \,|_0^1 - \int_0^1 Du_2D(\alpha(x)D^2u_1)\,dx \\
&\quad -u_2D(\alpha(x)D^2u_1)\,|_0^1 + \int_0^1 Du_2D(\alpha(x)D^2u_1)\,dx \\
&= 0
\end{aligned} \tag{59}$$

where again the boundary terms are each zero.

Clearly $A_2$ is also dissipative in this case. But $A_2 = (-A_2)^*$ and thus $(-A)^*$ is dissipative. Now all of the theorems apply as in the case of constant $\alpha$ and guarantee the existence of a unique solution to the differential equation.

### *2.5 Constant coefficient Kelvin-Voigt damping*

Consider $u_{tt} + \beta u_{txxxx} + \alpha u_{xxxx} = 0$ with $u(0,x) = u_0$, $u_t(0,x) = u_{t0}$ with the boundary conditions as before. Throughout this section $\alpha$ is a positive constant. To begin,

$$
\begin{aligned}
&= \beta D u_2 D^2 u_2 \,|_0^1 - \beta \int_0^1 (D^2 u_2)^2 \, dx \\
&\leq 0
\end{aligned} \tag{64}
$$

since $\beta \geq 0$.

The next task is to identify $(-A_3)^*$.

$$
\left( -A_3 \begin{pmatrix} u_1 \\ u_2 \end{pmatrix}, \begin{pmatrix} v_1 \\ v_2 \end{pmatrix} \right) = \left( \begin{pmatrix} u_2 \\ -\alpha D^4 u_1 - \beta D^4 u_2 \end{pmatrix}, \begin{pmatrix} v_1 \\ v_2 \end{pmatrix} \right) \tag{65}
$$

$$
\begin{aligned}
&= \alpha \int_0^1 D^2 u_2 D^2 v_1 \, dx - \alpha \int_0^1 v_2 D^4 u_1 \, dx - \beta \int_0^1 v_2 D^4 u_2 \, dx \\
&= \alpha D u_2 D^2 v_1 \,|_0^1 - \alpha \int_0^1 D u_2 D^3 v_1 \, dx - \alpha v_2 D^3 u_1 \,|_0^1 \\
&\quad + \alpha \int_0^1 D v_2 D^3 u_1 \, dx - \beta v_2 D^3 u_2 \,|_0^1 + \beta \int_0^1 D v_2 D^3 u_2 \, dx \\
&= -\alpha u_2 D^3 v_1 \,|_0^1 + \alpha \int_0^1 u_2 D^4 v_1 \, dx + \alpha D v_2 D^2 u_1 \,|_0^1 \\
&\quad - \alpha \int_0^1 D^2 v_2 D^2 u_1 \, dx + \beta D v_2 D^2 u_2 \,|_0^1 - \beta \int_0^1 D^2 v_2 D^2 u_2 \, dx \\
&= -\alpha \int_0^1 D^2 v_2 D^2 u_1 \, dx + \alpha \int_0^1 u_2 D^4 v_1 \, dx - \beta D u_2 D^2 v_2 \,|_0^1 + \beta \int_0^1 D^3 v_2 D u_2 \, dx \\
&= -\alpha \int_0^1 D^2 v_2 D^2 u_1 \, dx + \alpha \int_0^1 u_2 D^4 v_1 \, dx + \beta u_2 D^3 v_2 \,|_0^1 - \beta \int_0^1 D^4 v_2 u_2 \, dx
\end{aligned}
$$

$$
= \left( \begin{pmatrix} u_1 \\ u_2 \end{pmatrix}, \begin{pmatrix} 0 & -1 \\ \alpha D^4 & -\beta D^4 \end{pmatrix} \begin{pmatrix} v_1 \\ v_2 \end{pmatrix} \right) \tag{66}
$$

Thus $(-A_3)^* = \begin{pmatrix} 0 & -1 \\ \alpha D^4 & -\beta D^4 \end{pmatrix}$.

Notice that this operator is not skew-adjoint. Hence, the operator $-A_3$ is not the generator of a group as in the previous cases.

It is necessary to check whether the operator $(-A_3)^*$ is dissipative. The details here follow as for $-A_3$ except for a few sign changes in the intermediate steps so that the result is the same, and consequently, the details are omitted.

It is also important that $-A_3$ be a closed operator. But the adjoint of a densely defined linear operator is always closed (see Theorem 25). Since, in the present case $-A_3 = ((-A_3)^*)^*$ and $(-A_3)^*$ is densely defined (same domain as $A_3$), then $-A_3$ is closed as desired.

The existence of a unique $C^1$ solution, $u(t)$, to (60) for any $u_0 \in D(A_3)$ is now guaranteed by Theorem 27 and Theorem 29.

### *2.6 Spatial dependence in the damping coefficient*

Consider $u_{tt} + D^2(\beta(x)D^2u_t) + \alpha D^4 u = 0$. Assume $\beta(x) \geq 0$ for all $x \in [0,1]$, and $\alpha$ a positive constant as before. Formulate the abstract system as before with the operator defined as follows:

$$A = A_4 = \begin{pmatrix} 0 & -1 \\ \alpha D^4 & D^2(\beta(\cdot)D^2) \end{pmatrix}. \tag{67}$$

Note that $D(A_4) = D(A_3)$. It will now be established that $-A_4$ is dissipative. The details follow the pattern of the case for constant $\beta$. Boundary terms are zero as before. The symbol $'$ and the symbol $D$ will both be used to represent differentiation with respect to the spatial variable. Let $u \in D(A_4)$ be given. Then,

$$\begin{aligned}
(-A_4 u, u) &= \left( \begin{pmatrix} 0 & 1 \\ -\alpha D^4 & -D^2(\beta(\cdot)D^2) \end{pmatrix} \begin{pmatrix} u_1 \\ u_2 \end{pmatrix}, \begin{pmatrix} u_1 \\ u_2 \end{pmatrix} \right) \\
&= \left( \begin{pmatrix} u_2 \\ -\alpha D^4 u_1 - D^2(\beta(\cdot)D^2 u_2) \end{pmatrix}, \begin{pmatrix} u_1 \\ u_2 \end{pmatrix} \right) \\
&= \alpha \int_0^1 u_2'' u_1''\, dx + \int_0^1 \left(-\alpha D^4 u_1 - D^2(\beta(x)D^2 u_2)\right) u_2\, dx \\
&= \alpha u_2' u_1'' \,|_0^1 - \alpha \int_0^1 u_2' u_1'''\, dx - \alpha u_2 u_1''' \,|_0^1 + \alpha \int_0^1 u_2' u_1'''\, dx \\
&\quad - u_2 D(\beta(x)D^2 u_2) \,|_0^1 + \int_0^1 u_2' D(\beta(x)D^2 u_2)\, dx \\
&= u_2' \beta(x) D^2 u_2 \,|_0^1 - \int_0^1 u_2'' \beta(x) u_2''\, dx \\
&= -\int_0^1 \beta(x)(u_2'')^2\, dx
\end{aligned} \tag{68}$$

and this is clearly not positive since $\beta(x) \geq 0$ for each $x$.

Next the adjoint $(-A_4)^*$ of $-A_4$ will be computed. Again the details follow as in the case for constant $\beta$. For $u \in D(A_4)$,

$$
\begin{aligned}
(-A_4u, v) &= \left( \begin{pmatrix} 0 & 1 \\ -\alpha D^4 & -D^2(\beta(\cdot)D^2) \end{pmatrix} \begin{pmatrix} u_1 \\ u_2 \end{pmatrix}, \begin{pmatrix} v_1 \\ v_2 \end{pmatrix} \right) \\
&= \left( \begin{pmatrix} u_2 \\ -\alpha D^4 u_1 - D^2(\beta(\cdot)D^2 u_2) \end{pmatrix}, \begin{pmatrix} v_1 \\ v_2 \end{pmatrix} \right) \\
&= \alpha \int_0^1 u_2'' v_1''\, dx - \int_0^1 \left(\alpha D^4 u_1 + D^2(\beta(x)D^2 u_2)\right) v_2\, dx \\
&= \alpha u_2' v_1'' \,|_0^1 - \alpha \int_0^1 u_2' v_1'''\, dx - \alpha v_2 u_1''' \,|_0^1 + \alpha \int_0^1 v_2' u_1'''\, dx \\
&\quad - v_2 D(\beta(x)D^2 u_2) \,|_0^1 + \int_0^1 v_2' D(\beta(x)D^2 u_2)\, dx \\
&= -\alpha u_2 v_1''' \,|_0^1 + \alpha \int_0^1 u_2 D^4 v_1\, dx + \alpha v_2' u_1'' \,|_0^1 - \alpha \int_0^1 v_2'' u_1''\, dx \\
&\quad + v_2' \beta(x) u_2'' \,|_0^1 - \int_0^1 v_2'' \beta(x) u_2''\, dx \\
&= -\alpha \int_0^1 v_2'' u_1''\, dx + \alpha \int_0^1 u_2 D^4 v_1\, dx - v_2'' \beta(x) u_2'' \,|_0^1 + \int_0^1 u_2' D(\beta(x) v_2'')\, dx \\
&= -\alpha \int_0^1 v_2'' u_1''\, dx + \alpha \int_0^1 u_2 D^4 v_1\, dx + u_2 D(\beta(x) v_2'') \,|_0^1 - \int_0^1 u_2 D^2(\beta(x) v_2'')\, dx \\
&= \left( \begin{pmatrix} u_1 \\ u_2 \end{pmatrix}, \begin{pmatrix} 0 & -1 \\ \alpha D^4 & -D^2(\beta(\cdot)D^2) \end{pmatrix} \begin{pmatrix} v_1 \\ v_2 \end{pmatrix} \right) \\
&= (u, (-A_4)^* v). \qquad (69)
\end{aligned}
$$

Therefore,

$$
(-A_4)^* = \begin{pmatrix} 0 & -1 \\ \alpha D^4 & -D^2(\beta(\cdot)D^2) \end{pmatrix}. \qquad (70)
$$

The argument to show that $(-A_4)^*$ is dissipative follows the established pattern and has the desired result.

The domain, $D((-A_4)^*)$, of $(-A_4)^*$ is the same as $D(A_4)$. This is clear from an inspection of the operators. Hence $(-A_4)^*$ is densely defined. Note that $((-A_4)^*)^* = -A_4$ and thus $-A_4$ is closed by Theorem 25.

Existence of a $C^1$ solution in now assured by Theorem 27 and Theorem 29 as before.

### *2.7 Chapter summary*

Basic theorems of semigroup theory have been reviewed and the concept of abstract formulation of a differential equation has been discussed. A careful description of a linear space and a careful selection of operators have been provided to demonstrate the terminology. The hypotheses of an appropriate existence theorem were shown to be satisfied.

The beam equations for constant and spatially varying $\alpha$ have been considered. Also, for constant $\alpha$, the cases of constant and spatially varying $\beta$ have been considered. Existence and uniqueness of solutions has been established.

In the next chapter, consideration is given to a nonautonomous problem. All of the present chapter's concepts will be needed there. Additional concepts will also be introduced.

## III. *The nonautonomous problem*

In this chapter the coefficient of damping is allowed to vary with time. A suitable theory for this case will be presented. Combined temporal and spatial dependence is not difficult once the time dependent case is complete. The case of a time dependent $\alpha$ is also treated. This will require the introduction of an additional theorem.

### *3.1 An overview of the simple nonautonomous case*

Consider $u_{tt}+D^2(\beta(t)D^2u_t)+\alpha D^4u = 0$. Assume $\alpha > 0$ and $\beta(t) \geq 0$ for all $t \in [0,T]$. Furthermore, $\beta$ is assumed to be continuously differentiable. As before, formulate the problem as the abstract system

$$u_t + Au = 0;\ u(0) = u_0 \tag{71}$$

where now,

$$A = A(t) = A_5 = \begin{pmatrix} 0 & -1 \\ \alpha D^4 & \beta(t)D^4 \end{pmatrix} \tag{72}$$

where $D(A_5) = D(A_3)$. Note that for each $t$ this behaves the same as in the case of constant $\beta$. The arguments for the adjoint and dissipativity are not repeated.

Because of the explicit time dependence, (71) is called an evolution equation. For each $t > 0$, the previous arguments apply to establish that $-A_5$ is the generator of a $C_0$ semigroup of contractions which provides a solution as before. If $\beta(t)$ is suitably smooth it is reasonable to expect to be able to piece together, from the solutions for individual values of $t$, an overall solution. However, the theorems which gave solutions for individual values of $t$ do not guarantee the necessary behavior (*eg* continuity, differentiability) for their composite to be a solution of the evolution equation.

The solution strategy is as follows. At $t = 0$ the state of the system is specified. The solution of the equation for $\beta(0)$ is some surface, as in Figure 2. But, since $\beta$ changes with time, this solution is only accurate for small values of $t$. Suppose $T$, the largest value

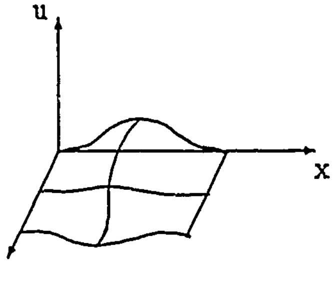


Figure 2. The initial solution

of interest, is larger than the interval for which the solution based on $\beta(0)$ is sufficiently accurate. Then an iterative procedure is used. This will be briefly outlined.

Let $0 = t_0 < t_1 < t_2 < \cdots < t_n = T$ describe a partition of $[0, T]$. Apply the results of the earlier work to the constant coefficient case corresponding to $\beta(0)$. Let $\hat{u}(t) = S_0(t - t_0)u_0$ on $[t_0, t_1]$, where $S_0$ is the semigroup generated by $-A_5(0)$. Then $\hat{u}(t_1)$ is an estimate of $u(t_1)$. Apply the earlier work again, this time for the constant coefficient $\beta(t_1)$. This gives $\hat{u}(t) = S_1(t - t_1)\hat{u}(t_1)$ on $[t_1, t_2]$. Take $\hat{u}(t_2)$ as an approximate value for $u(t_2)$. If $\beta(t)$ is appropriately smooth then a solution with any specified accuracy is possible by choosing a small enough upper bound on $max\{t_j - t_{j-1} | \; j = 1, 2, \cdots, n\}$. Conceptually, this is like Euler's method in numerical analysis.

In this section, necessary conditions for the solution strategy to make sense are identified. This will be preceeded by appropriate introductions of additional terminology and theorems.

In the earlier solution strategy a semigroup $S(t)$ was obtained from its infinitesimal generator $-A_3$. But $A_5 = A(t)$ depends on $t$ and there is some risk of confusion of parameters. The semigroup generated by $-A(t)$ will be denoted $S_t(s)$. Suppose now that a limiting operator, which always uses the current element of the semigroup, is desired. This might be represented by something like $S_s(s)$, but this would certainly be confusing. An operator is needed which can propogate from some time $s$ to some time $t$ while using the appropriate element of the semigroup at each instant of time.

The common choice of notation for this situation is $U$ with two parameters. Namely, $U(t, s)$ is used for the desired operator, when it exists. The term *evolution system* is used

to refer to $U(t,s)$. (Some authors use the term *evolution operator.*) So, while $U$ and $S$ play roles which are conceptually very similar, the terminology is quite different. The following standard definition is taken from [58:pg 129].

**Definition 33** *A two parameter family of bounded linear operators $U(t,s)$, $0 \leq s \leq t \leq T$, on $X$ is called an* evolution system *if the following two conditions are satisfied:*

1. $U(s,s) = I$, $U(t,r)U(r,s) = U(t,s)$ for $0 \leq s \leq r \leq t \leq T$.
2. $(t,s) \rightarrow U(t,s)$ is strongly continuous for $0 \leq s \leq t \leq T$.

When there is an evolution system corresponding to a problem with a time dependent operator, the evolution system is used to produce a solution in the same way a semigroup is used for the autonomous case. That is, $u(t) = U(t,0)u_0$.

Notice that property 1 of an evolution system (Definition 33) says that small steps and large steps give the same result as long as the ultimate end points are the same.

With $A_5$ allowed to depend on $t$, the same basic ideas as before are still applicable. It is necessary, however, to modify the requirements on $A_5$ to assure existence of a solution. Of course some smoothness of the map $t \rightarrow A(t)$ is required. The map is required to be continuous and also the concept of a stable family $\{A(t)\}_{t\in[0,T]}$ is introduced. Further, since $A_5$ is allowed to change as $t$ changes, there is the possibility, for an iterative scheme, that the image of $A_5(t_1)$ would fail to be in the domain of $A_5(t_2)$ (see Figure 3). This would cause the iterative scheme to fail. It is useful to identify an appropriate subspace $Y$, $Y \subset X$, such that for all $t \in [0,T]$, it happens that $Y \subset D(A(t))$. In this regard the notion of $Y$ being *$A(t)$-admissible* is presented. Such a $Y$ will not be very meaningful unless it is dense in $X$. This will be required.

When a solution is obtained in $Y$, it is termed a $Y$-valued solution [58:pp 139-140]. This is not a significant restriction for the application under consideration since $Y$ will be all of $D(A)$. However, the properties of $Y$-valued solutions are useful.

**Definition 34** *A function $u \in C([0,T] : Y)$ is a $Y$-valued solution of the initial value problem (71) if $u \in C^1((0,T] : X)$ and (71) is satisfied in $X$.*

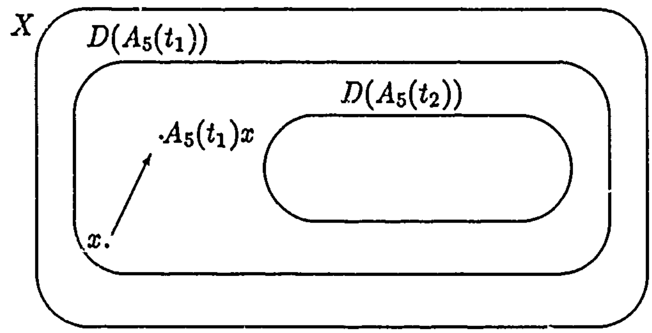


Figure 3. The different domains

Theory sufficient to establish the existence of a unique solution is presented in the next section. The development is gradual in that several of the theorems will be minor extensions of the immediately preceeding theorem. While this means there will be some overlap, it also has the advantage of step by step development. Most of the proofs will be by citation only. That is, while all the theorems are available, is seems prudent to gather statements of the theorems together here. It is not necessary to reproduce all the proofs.

### *3.2 The simple nonautonomous case*

The specific problem introduced in (71) and (72) is now addressed.

#### *3.2.1 Some technical preliminaries*

A fundamental concept is that of a stable family [58:pg 130].

**Definition 35** *A family $\{A(t)\}_{t\in[0,T]}$ of infinitesimal generators of $C_0$ semigroups on a Banach space $X$ is called stable if there are constants $M \geq 1$ and $\omega$ (called stability constants) such that*

$$\rho(A(t)) \supset (\omega, \infty) \text{ for all } t \in [0, T] \tag{73}$$

*and*

$$\left\| \prod_{j=1}^{k} R(\lambda : A(t_j)) \right\| \leq M(\lambda - \omega)^{-k} \tag{74}$$

*for $\lambda > \omega$ and every finite sequence $0 \leq t_1 \leq t_2 \leq \cdots \leq t_k \leq T; k = 1, 2, \cdots$.*

For $k = 1$ this is just the resolvent condition for an individual operator $A$ to be the generator of a $C_0$ semigroup [58:pg 12, Corollary 1.3.8]. Stability of the family essentially means that the bounds on the resolvents of individual operators $A(t_j)$ can be combined to give a bound on the composition of resolvents of $A(t_j)$'s. There are other conditions equivalent to (74), *eg* [58:pg 131, Theorem 5.2.2].

The next theorem is a standard, very powerful, result.

**Theorem 36 (Hille-Yosida)** *A linear operator $A$ is the infinitesimal generator of a $C_0$ semigroup of contractions $S(t)$, $t \geq 0$, if and only if $A$ is closed, $\overline{D(A)} = X$, and the resolvent set $\rho(A)$ of $A$ contains $\Re^+$, and for every $\lambda > 0$*

$$\|R(\lambda : A)\| \leq 1/\lambda. \tag{75}$$

**Proof:** See [58:pp 8-11]. □

There will be several occasions to use the following criterion to establish that a particular family is stable.

**Lemma 37** *If, for each $t \in [0, T]$, $A(t)$ generates a $C_0$ semigroup of contractions, then the family is stable.*

**Proof:** This is a straightforward application of Definition 35. For contraction semigroups it is appropriate to choose $\omega = 0$. Then, for each $t_j$, appeal to Theorem 36 to bound the resolvent by $1/\lambda$. Now it is clearly suitable to choose $M = 1$. □

Now the phrase A-admissible will be defined [58:pg 122].

**Definition 38** *Let $S(t)$ be a $C_0$ semigroup and let $A$ be its infinitesimal generator. A subspace $Y$ of $X$ is called $A$-admissible if it is an invariant subspace of $S(t)$, $t \geq 0$, and the restriction of $S(t)$ to $Y$ is a $C_0$ semigroup in $Y$ (ie, it is strongly continuous in the norm $\|\cdot\|_Y$).*

### 3.2.2 Key theorems

A first step toward finding the evolution system for (71) is given in the following theorem [58:pg 135]. The space $Y$ in the theorem has not yet been chosen. When the choice is made, it will be with these hypotheses in mind. The points of $Y$ will come from $X$ and the restriction, of an operator $A$, to $Y$ will be called the part of $A$ in $Y$ and be denoted by $\tilde{A}$. When it is time to apply these th rems, the place of $A$ will be taken by $-A_5$ defined in (72).

**Theorem 39** *Let $A(t)$, $0 \leq t \leq T$, be the infinitesimal generator of a $C_0$ semigroup $S_t(s)$, $s \geq 0$, on $X$. Suppose the family $\{A(t)\}_{t\in[0,T]}$ satisfies the conditions*

1. $\{A(t)\}_{t\in[0,T]}$ is a stable family with stability constants $M, \omega$.
2. $Y$ is $A(t)-$admissible for $t \in [0, T]$ and the family $\{\tilde{A}(t)\}_{t\in[0,T]}$ of parts $\tilde{A}(t)$ of $A(t)$ in $Y$, is a stable family in $Y$ with stability constants $\tilde{M}, \tilde{\omega}$.
3. For $t \in [0, T]$, $D(A(t)) \supset Y$, $A(t)$ is a bounded operator from $Y$ into $X$ and $t \to A(t)$ is continuous in the $B(Y, X)$ norm $\|\cdot\|_{Y\to X}$.

*Then, there exists a unique evolution system $U(t, s), 0 \leq s \leq t \leq T$, in $X$ satisfying*

1. $\|U(t, s)\| \leq Me^{\omega(t-s)}$ for $0 \leq s \leq t \leq T$.
2. $\frac{\partial^+}{\partial t}U(t, s)v \mid_{t=s}= A(s)v$ for $v \in Y,\ 0 \leq s \leq T$.
3. $\frac{\partial}{\partial s}U(t, s)v = -U(t, s)A(s)v$ for $v \in Y,\ 0 \leq s \leq t \leq T$.

*The derivative from the right in the second item and the derivative in the third are in the strong sense in $X$.*

**Proof:** See [58:pg 135-138]. □

The properties of $U$ established in the theorem are useful in demonstrating the uniqueness of the candidate solution it generates. The portion of the proof which describes the construction of the evolution system is repeated here for convenience and because of its role in the construction of a solution.

Consider an approximation of the family $\{A(t)\}_{t\in[0,T]}$ by piecewise constant families $\{A_n(t)\}_{t\in[0,T]}$, $n = 1, 2, \ldots$, defined as follows: Let $t_k^n = (k/n)T$, $k = 0, 1, \ldots, n$ and let

$$\begin{cases} A_n(t) &= A(t_k^n) \text{ for } t_k^n \leq t < t_{k+1}^n,\ k = 0, 1, \ldots, n-1 \\ A_n(T) &= A(T). \end{cases} \tag{76}$$

Since $t \to A(t)$ is continuous in the $B(Y, X)$ norm it follows that

$$\|A(t) - A_n(t)\|_{Y\to X} \to 0 \ as\ n \to \infty \tag{77}$$

uniformly in $t \in [0, T]$. From the definition of $A_n(t)$ and the hypotheses of Theorem 39 it follows readily that for $n \geq 1, \{A_n(t)\}_{t\in[0,T]}$ is a stable family in $X$ with constants $M, \omega$ while $\{\tilde{A}_n(t)\}_{t\in[0,T]}$ is a stable family in $Y$ with constants $\tilde{M}, \tilde{\omega}$.

Next, for each $n$ define a two parameter family of operators $U_n(t, s)$, $0 \leq s \leq t \leq T$ by,

$$U_n(t,s) = \begin{cases} S_{t_j^n}(t-s) \text{ for } t_j^n \leq s \leq t \leq t_{j+1}^n \\ S_{t_k^n}(t - t_k^n)\left[\prod_{j=l+1}^{k-1} S_{t_j^n}\left(\frac{T}{n}\right)\right] S_{t_l^n}(t_{l+1}^n - s) \\ \qquad \text{for } k > l,\ t_k^n \leq t \leq t_{k+1}^n,\ t_l^n \leq s \leq t_{l+1}^n . \end{cases} \tag{78}$$

It is straightforward to verify that $U_n(t, s)$ is an evolution system. Then, let

$$U(t,s)x = \lim_{n\to\infty} U_n(t,s)x \text{ for } x \in X,\ 0 \leq s \leq t \leq T. \tag{79}$$

Details of the proof that this limit exists and is an evolution system are in [58:pg 135-138].

Now consider the associated uniqueness result.

**Theorem 40** *Let $\{-A(t)\}_{t\in[0,T]}$ be a family of infinitesimal generators of $C_0$ semigroups on $X$ satisfying the conditions of Theorem 39. If the initial value problem (71) has a Y-valued solution $u$, then this solution is unique; and moreover, it is given by*

$$u(t) = U(t,s)v \tag{80}$$

*where $U(t,s)$ is the evolution system provided by Theorem 39 and $v$ is the value of $u$ at time $t = s$.*

**Proof:** See [58:pg 140]. □

The next theorem completes the list of properties required of $U$ in order to guarantee existence of a solution. In its proof, the need for results so far obtained becomes clear.

**Theorem 41** *Let $\{A(t)\}_{t\in[0,T]}$ satisfy the conditions of Theorem 39 and let $U(t,s)$, $0 \leq s \leq t \leq T$ be the evolution system given in Theorem 39. If*

1. $U(t,s)Y \subset Y$ for $0 \leq s \leq t \leq T$ and
2. For $v \in Y$, $U(t,s)v$ is continuous in $Y$ for $0 \leq s \leq t \leq T$

*then for every $v \in Y$, $U(t,s)v$ is the unique $Y$−valued solution of the initial value problem*

$$du(t)/dt = A(t)u(t) \text{ for } 0 \leq s \leq t \leq T \tag{81}$$

$$u(s) = v. \tag{82}$$

**Proof:** See [58:pg 141]. □

### *3.2.3 A strengthened hypothesis*

An alternative for the second hypothesis of Theorem 39 is now presented. It will be referred to as condition $2^+$. This condition appears in [35:pg 138]. It will be used to establish the additional hypotheses of Theorem 41.

($2^+$) There is a family $\{Q(t)\}_{t\in[0,T]}$ of isomorphisms ([54:pg 173] or [65:pp184, 199]) of $Y$ onto $X$ such that for every $v \in Y$, $Q(t)v$ is Lipschitz continuous in $X$ on $t \in [0, T]$ and

$$Q(t)A(t)Q(t)^{-1} = A(t) + B(t) \tag{83}$$

where $B(t)$, $0 \leq t \leq T$, is a strongly continuous family of bounded operators on $X$.

The next lemma establishes that this is indeed at least as strong a condition as hypothesis 2 of Theorem 39.

**Lemma** ~~.~~ *Hypothesis 1 of Theorem 39 and condition* $2^+$ *imply hypothesis 2 of Theorem 39.*

**Proof:** See [58:pg 142]. The condition $2^+$ is not as strong as the corresponding condition in the reference. Yet, the proof carries over with no essential modification. Also, see comments in Appendix D. □

Next it will be shown that the condition $2^+$ is, in fact, stronger than hypothesis 2 of Theorem 39. A preliminary lemma is appropriate.

**Lemma 43** *Let U(t,s),* $0 \leq s \leq t \leq T$ *be an evolution system in a Banach space* $X$ *satisfying* $\|U(t,s)\| \leq M$ *for* $0 \leq s \leq t \leq T$. *If H(t) is a family of integrable linear operators in* $X$ *such that for almost every* $t$, $\|H(t)\| < H < \infty$, *then there exists a unique family of bounded linear operators V(t,s),* $0 \leq s \leq t \leq T$ *on* $X$ *such that*

$$V(t,s)x = U(t,s)x + \int_s^t V(t,r)H(r)U(r,s)x\,dr \quad for \quad x \in X \tag{84}$$

*and V(t,s)x is continuous in s,t for* $0 \leq s \leq t \leq T$.

**Proof:** The proof is standard for Volterra integrals of the second kind. For details see [58:pp 142-143]. The hypothesis of integrability is replaced, in the reference, with strong continuity. But, with the bound $H$ described in the hypothesis, the cited proof holds with no essential modification. □

**Theorem 44** *Let* $A(t)$, *for* $0 \leq t \leq T$, *be the infinitesimal generator of a* $C_0$ *semigroup on* $X$. *If the family* $\{A(t)\}_{t\in[0,T]}$ *satisfies the conditions 1 and 3 of Theorem 39 and condition* $2^+$, *then there exists a unique evolution system* $U(t,s)$, $0 \leq s \leq t \leq T$, *in* $X$ *satisfying the following 5 conditions:*

1. $\|U(t,s)\| \leq Me^{\omega(t-s)}$ for $0 \leq s \leq t \leq T$,
2. $\frac{\partial^+}{\partial t}U(t,s)v \mid_{t=s}= A(s)v$ for $v \in Y$, $0 \leq s \leq T$,
3. $\frac{\partial}{\partial s}U(t,s)v = -U(t,s)A(s)v$ for $v \in Y$, $0 \leq s \leq t \leq T$,
4. $U(t,s)Y \subset Y$ for $0 \leq s \leq t \leq T$, and

5. For $v \in Y$, $U(t,s)v$ is continuous in $Y$ for $0 \leq s \leq t \leq T$.

**Proof:** See [58:pp 143-145]. Some comment on the cited proof is appropriate since the condition $2^+$ in this work is weaker than the corresponding condition in the reference. In the reference, $Q$ is required to be continuously differentiable. Then $\dot{Q}$ represents the derivative. In this work, $Q$ is only required to be Lipschitz continuous. But Lipschitz continuity implies absolute continuity which implies that there is some integrable function, which will also be denoted $\dot{Q}$, such that

$$Q(t) = Q(0) + \int_0^t \dot{Q}(\tau)\,d\tau. \tag{85}$$

A similar result is obtained in [38:pp 505-507].

It follows that $Q$ is differentiable a.e. with the derivative given by $\dot{Q}$ wherever the derivative exists.

The boundedness of $Q^{-1}$ and the differentiability of $Q$ a.e. lead to the differentiability of $Q^{-1}$ a.e.. Where it exists, the derivative is given by

$$\frac{d}{dt}\left(Q(t)^{-1}x\right) = -Q(t)^{-1}\dot{Q}(t)Q(t)^{-1}x. \tag{86}$$

With these observations, the proof goes through as cited. □

These results can be combined as a corollary, as in [58:pg 145].

**Corollary 45** *Let $\{A(t)\}_{t\in[0,T]}$ be a family of infinitesimal generators of $C_0$ semigroups on $X$. If $\{A(t)\}_{t\in[0,T]}$ satisfies the hypotheses of Theorem 44 for every $v \in Y$ then the initial value problem*

$$du(t)/dt = A(t)u(t) \text{ for } s < t \leq T, \tag{87}$$

$$u(s) = v, \tag{88}$$

*has a unique $Y$-valued solution, $u$, on $s \leq t \leq T$.*

### *3.2.4 The existence theorem for solving (71)*

The problem of solving (71) will fall under the following special case of Corollary 45. The theorem is to be applied to $-A_5$, as discussed previously. The next theorem is a special case of Theorem 5.4.8 in [58:pg 145]. See also [31:pg 252].

**Theorem 46** *Let $\{A(t)\}_{t\in[0,T]}$ be a stable family of infinitesimal generators of $C_0$ semigroups on $X$. If $D(A(t)) = D$ is independent of $t$ and for $v \in D$, $A(t)v$ is Lipschitz continuous in $X$ then there exists a unique evolution system $U(t,s)$, $0 \le s \le t \le T$, satisfying the five results of Theorem 44 where $Y$ is the set $D$ equipped with the norm $\|v\|_Y = \|v\|_X + \|A(0)v\|_X$.*

**Proof:** See [58:pp 145-146]. Also see Appendix D which elaborates on one portion of the proof cited. □

It may be noted that Theorem 46 made no mention of condition $2^+$. The hypotheses of Theorem 46 are such that $Q(t) = I + A(t)$ is an acceptable choice. Thus, whenever Theorem 46 applies, there is no need to explicitly identify the isomorphism $Q$. In the following application, the choice $Q = I - (-A_5) = I + A_5$ will be appropriate.

### *3.2.5 An application of Theorem 46 to the case of nonautonomous damping*

The hypotheses of Theorem 46 are satisfied for the problem represented by (71) and (72). Theorem 46 is to be applied to $-A_5$. The satisfaction of the hypotheses will be considered in some detail. Note that $D(A_5) = D(A_3)$.

Let $Y$ be the same set of points as $D(A_5)$ but with the graph norm of $A_5(0)$

$$\|y\|_Y = \|y\|_X + \|A_5(0)y\|_X. \tag{89}$$

**Lemma 47** *The linear space $Y$ is a Banach space under $\|\cdot\|_Y$.*

**Proof:** Since it is easy to see that $Y$ is a normed linear space, all but completeness is clear. Let $\{y_n\}$ be a Cauchy sequence in $Y$. It is immediate that $\{y_n\}$ and $\{A_5(0)y_n\}$ are Cauchy sequences in $X$. Hence, there exist $y, \hat{y} \in X$ such that $y_n \to y$ and $A_5(0)y_n \to \hat{y}$.

Since $-A_5(0)$ is the generator of a $C_0$ semigroup of contractions (as shown in the case of $\beta$ a constant) it follows that $A_5(0)$ is a closed operator (see Theorem 36). (The closedness

of $-A_5(0)$, as an operator in $X$, was established in showing that it is the generator of a $C_0$ semigroup.) From the closedness of $A_5(0)$ it follows that $y \in D(A_5)$, and hence $y \in Y$. Thus $Y$ is complete. □

For each $t \in [0,T]$ it is clear that $-A_5(t)$ is the generator of a $C_0$ semigroup of contractions (from the case for constant $\beta$). Hence, by Lemma 37, the family is stable as required. The domain has no time dependence as is easily seen from its definition.

Finally, it is necessary to verify the Lipschitz continuity of $A_5$ with respect to $t$. Let $v \in D$ be given. First consider a straightforward approach to continuity. Let $\epsilon > 0$ be given. Identify $\delta > 0$ such that

$$| t_1 - t_2 | < \delta \Rightarrow \|A_5(t_1)v - A_5(t_2)v\|_X < \epsilon. \tag{90}$$

From the definitions

$$\begin{aligned} \|A_5(t_1)v - A_5(t_2)v\|_X &= \left\| \begin{pmatrix} 0 & 0 \\ 0 & (\beta(t_1) - \beta(t_2))D^4 \end{pmatrix} \begin{pmatrix} v_1 \\ v_2 \end{pmatrix} \right\|_X \\ &= \left\| \begin{pmatrix} 0 \\ (\beta(t_1) - \beta(t_2))D^4 v_2 \end{pmatrix} \right\|_X \\ &= | \beta(t_1) - \beta(t_2) | \left( \int_0^1 (D^4 v_2)^2 dx \right)^{1/2} \end{aligned} \tag{91}$$

Now, $v_2 \in H^4$ and $\beta \in C^0$ makes clear the existence of suitable $\delta > 0$. Specifically, $v_2 \in H^4 \Rightarrow \exists M$ such that $\|v_2\|_{H^4} \leq M$. Also, continuity of $\beta$ guarantees that there is some $\delta$ such that

$$| t_1 - t_2 | < \delta \Rightarrow | \beta(t_1) - \beta(t_2) | < \epsilon / M\,. \tag{92}$$

This is the required $\delta$.

In fact, whatever smoothness $\beta$ has will carry over to $A_5(t)v$. For example, to identify $\frac{d}{dt}A_5(t)v$, consider

$$\begin{aligned}
\lim_{h\to 0}\frac{A_5(t+h)v - A_5(t)v}{h} &= \lim_{h\to 0}\left(\frac{1}{h}\begin{pmatrix} 0 & 0 \\ 0 & (\beta(t+h)-\beta(t))D^4 \end{pmatrix}\begin{pmatrix} v_1 \\ v_2 \end{pmatrix}\right) \\
&= \lim_{h\to 0}\left(\frac{1}{h}\begin{pmatrix} 0 \\ (\beta(t+h)-\beta(t))D^4 v_2 \end{pmatrix}\right) \\
&= \lim_{h\to 0}\begin{pmatrix} 0 \\ \frac{\beta(t+h)-\beta(t)}{h}D^4 v_2 \end{pmatrix} \\
&= \begin{pmatrix} 0 \\ \beta' D^4 v_2 \end{pmatrix}. \qquad (93)
\end{aligned}$$

Since $\beta'$ is continuous and $D^4 v_2$ does not depend on $t$ it is now clear that $A_5(t)v$ is continuously differentiable with respect to $t$ for $t \in [0,T]$. Thus, in this particular application, a stronger condition than Lipschitz continuity is satisfied.

This completes the verification of hypotheses for the application of Theorem 46 to (71).

A brief review of what is known about solving (71), as a result of satisfying the hypotheses of Theorem 46, is appropriate.

1. Theorem 46 guarantees the existence of $U(t,s)$ satisfying the five results of Theorem 44.
2. Corollary 45 says the problem has a unique $Y$-valued solution.
3. Theorem 41 says $U(t,0)u_0$ is the unique $Y$-valued solution of the equation.
4. Theorem 39 (proof) describes the construction of a sequence $\{U_n\}$ whose limit is $U$.

Since $U(t,s)$ is bounded, the continuous dependence on the initial condition vector follows as in Theorem 31.

### 3.3 *Combined $t, x$ dependence in the damping coefficient*

Consider $u_{tt} + D^2(\beta(t,x)D^2u_t) + \alpha D^4 u = 0$. Require the constant $\alpha > 0$ and $\beta(t,x) \geq 0$ as before. Furthermore, require $\beta$ and $\frac{\partial\beta}{\partial t}$ to be twice continuously differentiable with respect to $x$, and continuously differentiable with respect to $t$. The problem can be formulated as a system as in (71) with

$$A_6 = \begin{pmatrix} 0 & -1 \\ \alpha D^4 & D^2(\beta(t,\cdot)D^2) \end{pmatrix}. \tag{94}$$

Note that $D(A_6) = D(A_3)$.

It is not difficult to see that the hypotheses of Theorem 46 are again satisfied. Indeed, for each $t$, $-A_6$ generates a $C_0$ semigroup of contractions as in the $\beta(x)$ case. Hence, the family $\{-A_6(t)\}_{t\in[0,T]}$ is stable. Furthermore, note that $D(-A_6)$ does not depend on $t$. Also, $-A_6(t,\cdot) \in C^1$ as long as $\beta$ is appropriately smooth, as is required. Some detail is appropriate for this last point.

The computation which parallels (91) is now presented.

$$\begin{aligned}
\|A_6(t_1)v - A_6(t_2)v\|_X &= \left\| \begin{pmatrix} 0 & 0 \\ 0 & D^2\left(\beta(t_1,x) - \beta(t_2,x)\right)D^2 \end{pmatrix} \begin{pmatrix} v_1 \\ v_2 \end{pmatrix} \right\|_X \\
&= \left\| \begin{pmatrix} 0 \\ D^2\left([\beta(t_1,x) - \beta(t_2,x)]D^2 v_2\right) \end{pmatrix} \right\|_X \\
&= \left( \int_0^1 \left[ D^2\left( [\beta(t_1,x) - \beta(t_2,x)]D^2 v_2 \right) \right]^2 dx \right)^{1/2} \\
&= \left( \int_0^1 \left[ D\left( [\beta(t_1,x) - \beta(t_2,x)]D^3 v_2 \right.\right.\right. \\
&\qquad \left.\left.\left. + \left( \frac{\partial}{\partial x}\beta(t_1,x) - \frac{\partial}{\partial x}\beta(t_2,x) \right) D^2 v_2 \right) \right]^2 dx \right)^{1/2} \\
&= \left( \int_0^1 \left[ \left(\beta(t_1,x) - \beta(t_2,x)\right) D^4 v_2 \right.\right. \\
&\qquad +2\left( \frac{\partial}{\partial x}\beta(t_1,x) - \frac{\partial}{\partial x}\beta(t_2,x) \right) D^3 v_2
\end{aligned}$$

$$
\begin{aligned}
&+\left(\frac{\partial^2}{\partial x^2}\beta(t_1,x)-\frac{\partial^2}{\partial x^2}\beta(t_2,x)\right)D^2v_2\Big]^2\,dx\Bigg)^{1/2} \\
\leq\ & 3^{1/2}\left(\int_0^1\Big[(\beta(t_1,x)-\beta(t_2,x))^2\left(D^4v_2\right)^2\right. \\
&+4\left(\frac{\partial}{\partial x}(\beta(t_1,x)-\beta(t_2,x))\right)^2\left(D^3v_2\right)^2 \\
&\left.+\left(\frac{\partial^2}{\partial x^2}(\beta(t_1,x)-\beta(t_2,x))\right)^2\left(D^2v_2\right)^2\Big]\,dx\right)^{1/2} \\
\leq\ & 3^{1/2}\Bigg[\left(\int_0^1[\beta(t_1,x)-\beta(t_2,x)]^2\left(D^4v_2\right)^2dx\right)^{1/2} \\
&+2\left(\int_0^1\left[\frac{\partial}{\partial x}(\beta(t_1,x)-\beta(t_2,x))\right]^2\left(D^3v_2\right)^2dx\right)^{1/2} \\
&+\left(\int_0^1\left[\frac{\partial^2}{\partial x^2}(\beta(t_1,x)-\beta(t_2,x))\right]^2\left(D^2v_2\right)^2dx\right)^{1/2}\Bigg] \quad (95)
\end{aligned}
$$

The key to continue is that $\frac{\partial\beta}{\partial t}$ is twice continuously differentiable with respect to $x$ on a closed and bounded set. Thus, there exists some $M_\beta$ such that

$$
\begin{aligned}
\sup_{(t,x)\in[0,T]\times[0,1]}\left(\frac{\partial}{\partial t}\beta\right) &\leq M_\beta \\
\sup_{(t,x)\in[0,T]\times[0,1]}\left(\frac{\partial}{\partial t}\left(\frac{\partial}{\partial x}\beta\right)\right) &\leq M_\beta \\
\sup_{(t,x)\in[0,T]\times[0,1]}\left(\frac{\partial}{\partial t}\left(\frac{\partial^2}{\partial x^2}\beta\right)\right) &\leq M_\beta. \quad (96)
\end{aligned}
$$

Then,

$$
\begin{aligned}
\beta(t_1,x)-\beta(t_2,x) &\leq M_\beta|t_1-t_2| \\
\frac{\partial}{\partial x}(\beta(t_1,x)-\beta(t_2,x)) &= \frac{\partial}{\partial x}\beta(t_1,x)-\frac{\partial}{\partial x}\beta(t_2,x) \\
&\leq \sup_{(t,x)\in[0,T]\times[0,1]}\left(\frac{\partial}{\partial t}\left(\frac{\partial}{\partial x}\beta\right)\right)|t_1-t_2| \\
&\leq M_\beta|t_1-t_2| \\
\frac{\partial^2}{\partial x^2}(\beta(t_1,x)-\beta(t_2,x)) &\leq M_\beta|t_1-t_2|. \quad (97)
\end{aligned}
$$

Then, to continue,

$$
\begin{aligned}
\|A_6(t_1)v - A_6(t_2)v\|_X &\le 3^{1/2}\Bigg[M_\beta|t_1-t_2|\left(\int_0^1 \left(D^4v_2\right)^2 dx\right)^{1/2} \\
&\quad +2M_\beta|t_1-t_2|\left(\int_0^1 \left(D^3v_2\right)^2 dx\right)^{1/2} \\
&\quad +M_\beta|t_1-t_2|\left(\int_0^1 \left(D^2v_2\right)^2 dx\right)^{1/2}\Bigg] \\
&\le 3^{1/2}M_\beta|t_1-t_2|\Bigg[\left(\int_0^1 \left(D^4v_2\right)^2 dx\right)^{1/2} \\
&\quad +2\left(\int_0^1 \left(D^3v_2\right)^2 dx\right)^{1/2}+\left(\int_0^1 \left(D^2v_2\right)^2 dx\right)^{1/2}\Bigg]. \qquad (98)
\end{aligned}
$$

From this point the argument which follows (91) applies as before.

The computation which parallels equation (93) is presented next.

$$
\begin{aligned}
\lim_{h\to 0}\frac{A_6(t+h,x)v - A_6(t,x)}{h} &= \lim_{h\to 0}\left(\frac{1}{h}\begin{pmatrix} 0 & 0 \\ 0 & D^2\left(\beta(t+h,x)-\beta(t,x)\right)D^2 \end{pmatrix}\begin{pmatrix} v_1 \\ v_2 \end{pmatrix}\right) \\
&= \lim_{h\to 0}\left(\frac{1}{h}\begin{pmatrix} 0 \\ D^2\left(\left(\beta(t+h,x)-\beta(t,x)\right)D^2v_2\right) \end{pmatrix}\right) \\
&= \lim_{h\to 0}\begin{pmatrix} 0 \\ D^2\left(\frac{\beta(t+h,x)-\beta(t,x)}{h}D^2v_2\right) \end{pmatrix} \\
&= \begin{pmatrix} 0 \\ D^2\left(\beta' D^2v_2\right) \end{pmatrix}. \qquad (99)
\end{aligned}
$$

Continuity with respect to $t$ follows from an inequality similar to (98).

Thus, Theorem 46 applies to give the evolution system in terms of which a solution to the problem is given as described in the $\beta(t)$ case.

*3.4 A more general theorem*

The most general theorem used in this work is Theorem 48. It is a specialized combination of Theorem 3.1 of [35:pp 169-170] and The. rem 1 of [24:pp 275-276]. The proof is patterned after their proofs.

The problem of interest is

$$u_t + A(t, u(t))u = 0; \quad u(0) = u_0. \tag{100}$$

The theorem identifies sufficient conditions to guarantee the existence of a unique solution. Numerous hypotheses are required to describe the setting. They are presented first. The proof will be presented immediately afterward.

Throughout the remainder of this chapter, the following hypotheses will be assumed.

1. Let $X$ be a real, reflexive, separable, Banach space with norm $\|\cdot\|_X$. Let $Y$ be a subset of $X$ which, when endowed with an appropriate norm, $\|\cdot\|_Y$, is itself a real, reflexive, separable, Banach space. Let $\|\cdot\|_Y \geq \|\cdot\|_X$.
2. Let $W$ be an open set in $Y$.
3. Let $Q(t, w)$ be a collection of isomorphisms of $Y$ onto $X$ for $(t, w) \in [0, T] \times W$. Assume $T > 0$.
4. There are real constants $\lambda_Q$, $\hat{\lambda}_Q$, and $\mu_Q$ such that $\|Q(t, w)\|_{Y \to X} \leq \lambda_Q$, $\|Q(t, w)^{-1}\|_{X \to Y} \leq \hat{\lambda}_Q$, and $\|Q(t, w) - Q(\hat{t}, \hat{w})\|_{Y \to X} \leq \mu_Q(|\ t - \hat{t}\ | + \|w - \hat{w}\|_X)$ for arbitrary $(t, w)$, $(\hat{t}, \hat{w}) \in [0, T] \times W$.
5. Let $N(X)$ be the collection of all norms on $X$, equivalent to the given one, $\|\cdot\|_X$. That is, $N(X) = \{\|\cdot\|_\mu : \|\cdot\|_\mu$ is equivalent to $\|\cdot\|_X$, where $\mu$ comes from some index set$\}$. Let a metric (this will be validated shortly) for $N(X)$ be given by

$$d(\|\cdot\|_\mu, \|\cdot\|_\nu) = log \sup_{0 \neq y \in X} max \left\{ \frac{\|y\|_\mu}{\|y\|_\nu}, \frac{\|y\|_\nu}{\|y\|_\mu} \right\}. \tag{101}$$

6. Let $N(t, w) : \ [0, T] \times W \to N(X)$ be a function satisfying $d(N(t, w), \|\cdot\|_X) \leq \lambda_N$, $d(N(t, w), N(\hat{t}, \hat{w})) \leq \mu_N(|\ t - \hat{t}\ | + \|w - \hat{w}\|_X)$ for fixed nonnegative $\lambda_N$, $\mu_N$

$\in \Re$ and arbitrary $(t, w)$, $(\hat{t}, \hat{w}) \in [0, T] \times W$. Let $X_{N(t,w)}$ denote the space $X$ with the norm $N(t, w)$.

7. Let $\{A(t, w) : (t, w) \in t \in [0, T] \times W\}$ be a family of operators, $A(t, w) \in B(Y, X)$. There are real positive constants $\lambda_A$ and $\mu_A$ such that for each $(t, w) \in [0, T] \times W$, the following hold:

$$A(t, w) \in G(X_{N(t,w)}, 1, 0)$$

$$\|A(t, w)\|_{Y \to X} \leq \lambda_A$$

$$\|A(t, w) - A(\hat{t}, \hat{w})\|_{Y \to X} \leq \mu_A \left( |t - \hat{t}| + \|w - \hat{w}\|_X \right)$$

8. $Q(t, w)A(t, w)Q(t, w)^{-1} = A(t, w)$ for each $(t, w) \in [0, T] \times W$.

**Theorem 48** *When the above eight hypotheses are satisfied, the following conclusion holds: For each $u(0) = u_0 \in W$, there is some $\hat{T} > 0$ and a unique solution $u$ to $u_t + A(t, u(t))u = 0$ with $u(0) = u_0$ such that $u \in C([0, \hat{T}]; W) \cap C^1([0, \hat{T}]; X)$.*

The metric in Hypothesis 5 appears in [24:pg 275].

### *3.5 Lemmas for use in the proof of Theorem 48*

The proof requires several technical results. It will be useful to begin with an overview of the ideas involved.

The solution which is to be obtained will be $Y$-valued. This means that only points in $Y$ will be considered for values of $u(t)$. A candidate solution can be thought of as a curve in the space $Y$ with initial point $u_0$. There is no guarantee that solutions can be propagated for long periods of time. Hence, consideration is given to candidate solutions which are supposed to be valid until some time denoted $\hat{T}$ which is not yet specified.

The proof then is roughly outlined in the following steps. First, a set $E$ is formed which contains the candidate solutions. Elements of $E$ are curves in $Y$ with initial point $u_0$. It will be established that the chosen set is a complete metric space under an appropriate metric. Second, it is shown for each fixed $v \in E$, that the family $\{A(t, v(t))\}$

has an associated evolution system (which generates a solution to the linear problem $u_t + A(t, v(t))u = 0$). Third, it is shown that the solution to the linear problem is, in fact, in $E$. Also, the mapping from $E$ to $E$ thus established is a contraction mapping. Then its fixed point is the desired solution.

There are many details between this outline and the completion of the proof. They will be presented now.

### 3.5.1 Preliminary lemmas

**Lemma 49** *The proposed metric for $N(X)$ in Hypothesis 5 is valid.*

**Proof:** Validation is straightforward from the definition, *eg* [54:pg 45], [66:pg 27], or [73:pg 4]. In particular, if $\|\cdot\|_\mu = \|\cdot\|_\nu$ then it is clear that $d(\|\cdot\|_\mu, \|\cdot\|_\nu) = 0$ since $log 1 = 0$. Also, since $\frac{\|y\|_\mu}{\|y\|_\nu}$ or its reciprocal will always be greater than or equal to one, it is clear from the properties of the $log$ function that $d(\|\cdot\|_\mu, \|\cdot\|_\nu) \geq 0$. Similarly, if $\|\cdot\|_\mu \neq \|\cdot\|_\nu$, then $d(\|\cdot\|_\mu, \|\cdot\|_\nu) > 0$. Symmetry is obvious. The triangle inequality is a little tedious but straightforward. The details follow.

$$
\begin{aligned}
& d(\|\cdot\|_\mu, \|\cdot\|_\nu) + d(\|\cdot\|_\nu, \|\cdot\|_o) = log \sup_{0 \neq y \in Y} max \left\{ \frac{\|y\|_\mu}{\|y\|_\nu}, \frac{\|y\|_\nu}{\|y\|_\mu} \right\} \\
& + log \sup_{0 \neq y \in Y} max \left\{ \frac{\|y\|_\nu}{\|y\|_o}, \frac{\|y\|_o}{\|y\|_\nu} \right\} \\
= \; & log \left( \sup_{0 \neq y \in Y} max \left\{ \frac{\|y\|_\mu}{\|y\|_\nu}, \frac{\|y\|_\nu}{\|y\|_\mu} \right\} \sup_{0 \neq y \in Y} max \left\{ \frac{\|y\|_\nu}{\|y\|_o}, \frac{\|y\|_o}{\|y\|_\nu} \right\} \right) \\
\geq \; & log \sup_{0 \neq y \in Y} \left( max \left\{ \frac{\|y\|_\mu}{\|y\|_\nu}, \frac{\|y\|_\nu}{\|y\|_\mu} \right\} max \left\{ \frac{\|y\|_\nu}{\|y\|_o}, \frac{\|y\|_o}{\|y\|_\nu} \right\} \right) \\
= \; & log \sup_{0 \neq y \in Y} \left( max \left\{ \frac{\|y\|_\mu}{\|y\|_\nu} \frac{\|y\|_\nu}{\|y\|_o}, \frac{\|y\|_\mu}{\|y\|_\nu} \frac{\|y\|_o}{\|y\|_\nu}, \frac{\|y\|_\nu}{\|y\|_\mu} \frac{\|y\|_\nu}{\|y\|_o}, \frac{\|y\|_\nu}{\|y\|_\mu} \frac{\|y\|_o}{\|y\|_\nu} \right\} \right) \\
= \; & log \sup_{0 \neq y \in Y} \left( max \left\{ \frac{\|y\|_\mu}{\|y\|_o}, \frac{\|y\|_\mu \|y\|_o}{\|y\|_\nu^2}, \frac{\|y\|_\nu^2}{\|y\|_\mu \|y\|_o}, \frac{\|y\|_o}{\|y\|_\mu} \right\} \right) \\
\geq \; & log \sup_{0 \neq y \in Y} max \left\{ \frac{\|y\|_\mu}{\|y\|_o}, \frac{\|y\|_o}{\|y\|_\mu} \right\} \\
= \; & d(\|\cdot\|_\mu, \|\cdot\|_o) \quad \square
\end{aligned} \tag{102}
$$

For use in the proof, a collection of possible solutions is formulated as a metric space. None of the candidates is allowed to go too far from the initial condition vector.

Let $u_0 \in W$ be given with $M = \|u_0\|_Y$. Choose $\hat{\rho} > 0$ with $\hat{\rho} < \infty$, such that $\hat{\rho} < dist(u_0, Y \backslash W)$. (The symbol '$\backslash$' is used for set subtraction.) It is advantageous in applications to choose $\hat{\rho}$ as large as possible within the bounds indicated. The set $Y \backslash W$ is nonempty since $W$ is an open, proper subset of $Y$.

**Definition 50** *Let $E$ be the set of all functions $v : [0, \hat{T}] \to Y$ such that for each $t, \hat{t} \in [0, \hat{T}]$, $\|v(t) - u_0\|_Y \leq \hat{\rho}$, and $\|v(t) - v(\hat{t})\|_X \leq \hat{L}|t - \hat{t}|$ where $\hat{L} = 2\lambda_A(R + M)\hat{\lambda}_Q\lambda_Q e^{\lambda_N}$ and the value of $\hat{T}$ is not yet specified.*

The purpose of these requirements is to keep $v(t)$ in $W$. The value of $\hat{\rho}$ has already been chosen. The choice of $\hat{T}$ is constrained by (148) and the discussion in the proof of Lemma 60. These are constraints which guarantee that a certain mapping, to be developed in the proof of the theorem, is a contraction mapping. Details concerning the choice of $\hat{T}$ will be provided after the mapping has been presented. But first, there are several more preliminaries.

Define a distance function for pairs of elements in $E$ as follows. For $v,\ w \in E$

$$d(v, w) = \sup_{0 \leq t \leq \hat{T}} \|v(t) - w(t)\|_X. \tag{103}$$

**Lemma 51** *The set $E$ with the distance function $d$ is a metric space.*

**Proof:** The verification that this is a legitimate metric is completely straightforward. It is easy to see that $d(v, v) = 0$. On the other hand, suppose $d(v, w) = 0$. Then for each $t \in [0, \hat{T}]$, $\|v(t) - w(t)\|_X = 0$. Since $\|\cdot\|_X$ is a norm, it follows that $v(t) = w(t)$ for each $t \in [0, \hat{T}]$. That is, $v = w$. When $v \neq w$, there is some $\hat{t}$ such that $v(\hat{t}) \neq w(\hat{t})$. Then

$$\begin{aligned} d(v, w) &\geq \sup_{t \in [0, \hat{T}]} \|v(t) - w(t)\|_X \\ &\geq \|v(\hat{t}) - w(\hat{t})\|_X \\ &> 0 \end{aligned} \tag{104}$$

since $\|\cdot\|_X$ is a norm. Symmetry is obvious. The triangle inequality is the only remaining issue. Let $u, v, w \in E$ be given. Then

$$\begin{aligned}
d(u,w) &= \sup_{t\in[0,\hat{T}]} \|u(t) - w(t)\|_X \\
&= \sup_{t\in[0,\hat{T}]} \|u(t) - v(t) + v(t) - w(t)\|_X \\
&\leq \sup_{t\in[0,\hat{T}]} \|u(t) - v(t)\|_X + \sup_{t\in[0,\hat{T}]} \|v(t) - w(t)\|_X \\
&= d(u,v) + d(v,w). \qquad (105)
\end{aligned}$$

Thus, $E$ is a metric space. □

It will be important for $E$ to be a complete metric space.

**Lemma 52** *The set $E$, with the metric defined in (103), is a complete metric space.*

**Proof:** Let $\{v_n\}$ be a Cauchy sequence in $E$. Then for each $t$, $\{v_n(t)\}$ is a Cauchy sequence in $X$ as is easily seen by examination of the metric on E. Since $X$ is a complete space, there is some $\hat{v}(t) \in X$ such that $v_n(t) \to \hat{v}(t)$ in the $X$ norm. It is easy to see that $\hat{v}$ is unique. It is not at all clear whether $\hat{v}$ lies in $Y$ or is an element of $E$. These issues are addressed next.

According to a standard theorem, (*eg* Theorem 7.70 of [20:pg 204]), $\{v_n(t)\}$ treated as a sequence in $Y$ has a weakly convergent subsequence, say $\{z_n(t)\}$. Then, (see Theorem 7.65 of [20:pg 202]) there is some $v \in Y$ such that $z_n \to v$. If $v = \hat{v}$ then no other subsequence could converge to any other point. But this is easily shown. Note that $z_n \to v$ in the $X$ norm. Thus $\|v(t) - \hat{v}(t)\|_X = 0$. This means that $v = \hat{v}$ pointwise. Thus $v$ is the only weak limit of $\{v_n\}$.

If $v$ satisfies the two r[illegible]ments on elements of $E$, then the argument for completeness will be finished. But,

$$\begin{aligned}
\|v(t) - u_0\|_Y &= \|v(t) - v_n(t) + v_n(t) - u_0\|_Y \\
&\leq \|v(t) - v_n(t)\|_Y + \|v_n(t) - u_0\|_Y \\
&\leq \epsilon + \hat{\rho} \qquad (106)
\end{aligned}$$

for any $\epsilon > 0$ when $n$ is sufficiently large. Thus the first requirement is satisfied. Furthermore,

$$\begin{aligned}
\left\|v(t) - v(\hat{t})\right\|_X &= \left\|v(t) - v_n(t) + v_n(t) - v_n(\hat{t}) + v_n(\hat{t}) - v(\hat{t})\right\|_X \\
&\leq \|v(t) - v_n(t)\|_X + \left\|v_n(t) - v_n(\hat{t})\right\|_X + \left\|v_n(\hat{t}) - v(\hat{t})\right\|_X \\
&\leq \epsilon/2 + \hat{L}|t - \hat{t}| + \epsilon/2 \qquad (107)
\end{aligned}$$

for any $\epsilon > 0$. Thus the second requirement is satisfied.

This completes the argument for the completeness of the metric space $E$. □

### *3.5.2 Preparing for an evolution system*

The proof of the theorem will require the solution of a sequence of linear equations. For each linear equation a solution is obtained from Theorem 46. In this subsection the linear equations are introduced and it is verified that they satisfy the hypothesis of Theorem 46.

Some convenient notation is now introduced. For each $v \in E$, let $N^v(t) = N(t, v(t)) = \|\cdot\|_t$, $Q^v(t) = Q(t, v(t))$, and $A^v(t) = A(t, v(t))$. Throughout the remainder of Section 3.5, $v$ will always represent an element of $E$.

Consider now, for fixed $v$, the (still) nonautonomous (but now) linear problem

$$\frac{du}{dt} + A^v(t)u = 0; \quad 0 \leq t \leq \hat{T}; \quad u(0) = u_0. \qquad (108)$$

The next major step is to establish the existence of an evolution system which solves this linear problem. The lemmas which follow are necessary to establish the hypotheses of a Theorem 46.

**Lemma 53** *The family $\{A^v(t)\} \subset G(X_{N(t,v(t))}, 1, 0)$ is stable with stability constants $M_{U_X} = e^{\lambda_N + 2\mu_N(1+\hat{L})\hat{T}}$, and $\omega = 0$.*

**Proof:** It is important to bound $\frac{\|\cdot\|_t}{\|\cdot\|_{\hat{t}}}$ for the given $v$ and arbitrary $t, \hat{t} \in [0, \hat{T}]$. First, consider that Hypothesis 6 and the second property of $E$ reveal a bound for the distance

between the two norms.

$$\begin{aligned} d(\|\cdot\|_t, \|\cdot\|_{\hat{t}}) &\leq \mu_N(|\, t - \hat{t}\,| + \|v(t) - v(\hat{t})\|_X) \\ &\leq \mu_N(|\, t - \hat{t}\,| + \hat{L}\,|\, t - \hat{t}\,|) \\ &= \mu_N(1 + \hat{L})\,|\, t - \hat{t}\,| \end{aligned} \tag{109}$$

From the definition of the distance function on the collection of norms (see Hypothesis 5) it follows that

$$log \sup_{0 \neq y \in X} max \left\{ \frac{\|y\|_t}{\|y\|_{\hat{t}}}, \frac{\|y\|_{\hat{t}}}{\|y\|_t} \right\} \leq \mu_N(1 + \hat{L})\,|\, t - \hat{t}\,|$$

$$\begin{aligned} \text{so that } \frac{\|y\|_t}{\|y\|_{\hat{t}}} &\leq \sup_{0 \neq y \in X} max \left\{ \frac{\|y\|_t}{\|y\|_{\hat{t}}}, \frac{\|y\|_{\hat{t}}}{\|y\|_t} \right\} \\ &\leq e^{\mu_N(1+\hat{L})|t-\hat{t}|} \end{aligned} \tag{110}$$

and the desired bound is $e^{\mu_N(1+\hat{L})|t-\hat{t}|}$.

A method from the proof of Proposition 3.4 of [31:pg 245] is used in the next step. From the definition of a stable family (Definition 35) and the equivalence of norms, write

$$\begin{aligned} \left\| \prod_{j=1}^{k} (A^v(t_j) + \lambda)^{-1} y \right\|_{\hat{T}} &\leq e^{\mu_N(1+\hat{L})(\hat{T}-t_k)} \left\| \prod_{j=1}^{k} (A^v(t_j) + \lambda)^{-1} y \right\|_{t_k} \\ &\leq e^{\mu_N(1+\hat{L})(\hat{T}-t_k)} \frac{1}{\lambda} \left\| \prod_{j=1}^{k-1} (A^v(t_j) + \lambda)^{-1} y \right\|_{t_k} \\ &\leq \frac{1}{\lambda} e^{\mu_N(1+\hat{L})(\hat{T}-t_k)} e^{\mu_N(1+\hat{L})(t_k - t_{k-1})} \left\| \prod_{j=1}^{k-1} (A^v(t_j) + \lambda)^{-1} y \right\|_{t_{k-1}} \\ &\vdots \\ &\leq \left(\frac{1}{\lambda}\right)^k e^{\mu_N(1+\hat{L})(\hat{T}-t_k)} e^{\mu_N(1+\hat{L})(t_k - t_{k-1})} \ldots e^{\mu_N(1+\hat{L})t_1} \|y\|_0 \\ &\leq \left(\frac{1}{\lambda}\right)^k e^{\mu_N(1+\hat{L})\hat{T}} \|y\|_0 \\ &\leq \left(\frac{1}{\lambda}\right)^k e^{\mu_N(1+\hat{L})(\hat{T}+\hat{T})} \|y\|_{\hat{T}} \end{aligned}$$

$$\leq \left(\frac{1}{\lambda}\right)^k e^{2\mu_N(1+\hat{L})\hat{T}} \|y\|_{\hat{T}}. \tag{111}$$

Clearly this bound holds for any of the norms. Note also that $\lambda = 1$ is a valid choice in the current setting.

This yields stability of the family $\{A^v(t)\}$ with stability constant $\widehat{M} = e^{2\mu_N(1+\hat{L})\hat{T}}$ for any $X_t$, meaning the set $X$ with any one of the norms $N(t, v(t))$. It remains to translate this to the original norm on $X$. But, from Hypotheses 5 and 6, it is easy to establish that $\|\cdot\|_X \leq e^{\lambda_N}\|\cdot\|_t$. Namely, from the definition in Hypothesis 5 and the bound in Hypothesis 6 it follows that

$$\begin{aligned} log \sup_{0 \neq y \in X} max \left\{ \frac{\|y\|_t}{\|y\|_X}, \frac{\|y\|_X}{\|y\|_t} \right\} &= d(\|\cdot\|_t, \|\cdot\|_X) \\ &\leq \lambda_N. \end{aligned} \tag{112}$$

Then

$$\sup_{0 \neq y \in X} max \left\{ \frac{\|y\|_t}{\|y\|_X}, \frac{\|y\|_X}{\|y\|_t} \right\} \leq e^{\lambda_N}. \tag{113}$$

Now it is clear that $\frac{\|y\|_X}{\|y\|_t} \leq e^{\lambda_N}$ and hence $\|y\|_X \leq e^{\lambda_N}\|y\|_t$ for any nonzero $y \in X$. Then $\{A^v(t)\}$ is a stable family in $X$ with stability constant $M_{U_X} = e^{\lambda_N}\widehat{M} = e^{\lambda_N + 2\mu_N(1+\hat{L})\hat{T}}$ as desired. □

**Lemma 54** *For each $v \in E$, the mapping $t \to A^v(t) \in B(Y, X)$ is Lipschitz continuous.*

**Proof:** The proof, which is straightforward, is outlined. Let $\hat{t}, t \in [0, \hat{T}], v \in E$ be given. Then

$$\begin{aligned} \|A^v(t) - A^v(\hat{t})\|_{Y \to X} &= \|A(t, v(t)) - A(\hat{t}, v(\hat{t}))\|_{Y \to X} \\ &\leq \mu_A \left(|t - \hat{t}| + \|v(t) - v(\hat{t})\|_X\right) \\ &\leq \mu_A \left(|t - \hat{t}| + \hat{L}|t - \hat{t}|\right) \\ &\leq \mu_A \left(1 + \hat{L}\right)|t - \hat{t}|. \quad \square \end{aligned} \tag{114}$$

**Lemma 55** *For each $t \in [0, \hat{T}]$, $v \in E$, the bound $\|Q^v(t)\|_{Y \to X} \leq \lambda_Q$ holds.*

### 3.5.3 The evolution system

The hypotheses of Theorem 46 are now satisfied and the theorem gives an evolution operator $U^v(t,s)$. This is a good time to review how the proof all fits together. Based on the previous discussion, an evolution operator can be obtained corresponding to $u_0$. If the image of elements of $W$, under the action of the operator $U^{u_0}$, is again in $E$, then an iterative procedure is justified. This subsection consists of preliminaries necessary to establish that such images are, in fact, again in $E$.

Now that the evolution operator has been identified, it will be useful to identify bounds for it. This will require attention to a portion of the proof of Theorem 44. Bounds are desired for $\|U^v\|_X$ and $\|U^v\|_Y$.

**Lemma 58** *For each $v \in E, \|U^v\|_X \le e^{\lambda_N + 2\mu_N(1+\hat{L})\hat{T}}$ .*

**Proof:** The bound for $\|U^v\|_X$ is immediate from Lemma 53 and property 1 of Theorem 44. In particular, property 1 of Theorem 44 says that $\|U^v(t,s)\|_X \le M_{U_X} e^{\omega(t-s)}$. Lemma 53 says $\omega = 0$ and $M_{U_X}$ has the value indicated. □

A bound for $\|U^v\|_Y$ is not as simple to obtain. The bound, which is established in the next lemma, will be denoted by $M_{U_Y}$.

**Lemma 59** *For each $v \in E$,*

$$\|U^v\|_Y \le \hat{\lambda}_Q \lambda_Q \exp\left[(\lambda_N + 2\mu_N(1+\hat{L})\hat{T}) + \mu_Q(1+\hat{L})\lambda_Q \hat{T} e^{(\lambda_N + 2\mu_N(1+\hat{L})\hat{T})}\right]. \tag{118}$$

**Proof:** The strategy is to define an intermediate operator $V^v$ such that $U^v = (Q^v)^{-1} V^v Q^v$. Notice that while $U^v : Y \to Y$ in this equation, the isomorphism $Q^v$ allows $V^v$ to be a mapping from $X$ to $X$. A bound on $\|V^v\|_X$ can then be multiplied by $\lambda_Q \hat{\lambda}_Q$ to give a bound for $\|U^v\|_Y$.

Let $C^v(t) = \dot{Q}^v(t)(Q^v(t))^{-1}$ where $\dot{Q}^v(t)$ is the same as discussed in conjunction with Theorem 44 with the choice $Q = I + A$. The intermediate operator $V^v$ will be defined in terms of $U^v : X \to X$. The operator $V^v$, here, is given by the $V$ of Lemma 43. From

**Proof:** This simply requires a careful look at the hypotheses. Let $t \in [0, \hat{T}]$, $v \in E$ be given. Then

$$\begin{aligned} \|Q^v(t)\|_{Y \to X} &= \|Q(t, v(t))\|_{Y \to X} \\ &\leq \lambda_Q \end{aligned} \tag{115}$$

where the inequality holds, for each $t$, by Hypothesis 4. Hence, since the bound is uniform, it also holds for the supremum over all $t$. $\square$

The case for $\|Q^v(t)^{-1}\|_{X \to Y} \leq \hat{\lambda}_Q$ is similar.

**Lemma 56** *The bound* $\|Q^v(t) - Q^v(\hat{t})\|_{Y \to X} \leq \mu_Q(1 + \hat{L}) \mid t - \hat{t} \mid$, *holds for* $\hat{t}, t \in [0, \hat{T}]$ *and each* $v \in E$.

**Proof:** The proof is a straightforward application of Hypothesis 4 and the definition of $E$.

$$\begin{aligned} \|Q^v(t) - Q^v(\hat{t})\|_{Y \to X} &= \|Q(t, v(t)) - Q(\hat{t}, v(\hat{t}))\|_{Y \to X} \\ &\leq \mu_Q(\mid t - \hat{t} \mid + \|v(t) - v(\hat{t})\|_X) \\ &\leq \mu_Q(\mid t - \hat{t} \mid + \hat{L} \mid t - \hat{t} \mid) \\ &= \mu_Q(1 + \hat{L}) \mid t - \hat{t} \mid . \qquad \square \end{aligned} \tag{116}$$

**Lemma 57** *For each* $t \in [0, \hat{T}]$ *and* $v \in E$, $Q^v(t) A^v(t) Q^v(t)^{-1} = A^v(t)$.

**Proof:** This is almost immediate from Hypothesis 8, which holds for each $t$.

$$\begin{aligned} Q^v(t) A^v(t) Q^v(t)^{-1} &= Q(t, v(t)) A(t, v(t)) Q(t, v(t))^{-1} \\ &= A(t, v(t)) \\ &= A^v(t). \qquad \square \end{aligned} \tag{117}$$

This completes the preliminaries necessary to obtain an evolution system corresponding to the linear equation 108.

*3.5.3 The evolution system*

The hypotheses of Theorem 46 are now satisfied and the theorem gives an evolution operator $U^v(t,s)$. This is a good time to review how the proof all fits together. Based on the previous discussion, an evolution operator can be obtained corresponding to $u_0$. If the image of elements of $W$, under the action of the operator $U^{u_0}$, is again in $E$, then an iterative procedure is justified. This subsection consists of preliminaries necessary to establish that such images are, in fact, again in $E$.

Now that the evolution operator has been identified, it will be useful to identify bounds for it. This will require attention to a portion of the proof of Theorem 44. Bounds are desired for $\|U^v\|_X$ and $\|U^v\|_Y$.

**Lemma 58** *For each* $v \in E, \|U^v\|_X \leq e^{\lambda_N + 2\mu_N(1+\hat{L})\hat{T}}$ .

**Proof:** The bound for $\|U^v\|_X$ is immediate from Lemma 53 and property 1 of Theorem 44. In particular, property 1 of Theorem 44 says that $\|U^v(t,s)\|_X \leq M_{U_X} e^{\omega(t-s)}$. Lemma 53 says $\omega = 0$ and $M_{U_X}$ has the value indicated. □

A bound for $\|U^v\|_Y$ is not as simple to obtain. The bound, which is established in the next lemma, will be denoted by $M_{U_Y}$.

**Lemma 59** *For each* $v \in E$,

$$\|U^v\|_Y \leq \hat{\lambda}_Q \lambda_Q \exp\left[(\lambda_N + 2\mu_N(1+\hat{L})\hat{T}) + \mu_Q(1+\hat{L})\lambda_Q \hat{T} e^{(\lambda_N + 2\mu_N(1+\hat{L})\hat{T})}\right]. \qquad (118)$$

**Proof:** The strategy is to define an intermediate operator $V^v$ such that $U^v = (Q^v)^{-1} V^v Q^v$. Notice that while $U^v : Y \to Y$ in this equation, the isomorphism $Q^v$ allows $V^v$ to be a mapping from $X$ to $X$. A bound on $\|V^v\|_X$ can then be multiplied by $\lambda_Q \hat{\lambda}_Q$ to give a bound for $\|U^v\|_Y$.

Let $C^v(t) = \dot{Q}^v(t)(Q^v(t))^{-1}$ where $\dot{Q}^v(t)$ is the same as discussed in conjunction with Theorem 44 with the choice $Q = I + A$. The intermediate operator $V^v$ will be defined in terms of $U^v : X \to X$. The operator $V^v$, here, is given by the $V$ of Lemma 43. From

Lemma 43, if $H(t) = C(t) = \dot{Q}(t)(Q(t))^{-1}$, then for any $x \in X$, $V^v$ is the unique solution of the integral equation

$$V^v(t,s)x = U^v(t,s)x + \int_s^t V^v(t,r)C(r)U^v(r,s)x\,dr. \tag{119}$$

It is shown in the proof of Theorem 44 that $V^v$, defined in this way, satisfies $U^v = (Q^v)^{-1}V^vQ^v$. For completeness, the relevant details are presented. (For convenience in writing, the superscripts '$v$' will be omitted.)

From the boundedness of $Q^{-1}$ and the differentiability of $Q$ it follows that $Q^{-1}$ is differentiable a.e.. Note that

$$\frac{d}{dt}\left(Q(t)^{-1}x\right) = -Q(t)^{-1}\dot{Q}(t)Q(t)^{-1}x. \tag{120}$$

As a temporary notational convenience, let $F(t,r) = U(t,r)Q(r)^{-1}$. Then

$$\begin{aligned}\frac{\partial}{\partial r}F(t,r)x &= \frac{\partial}{\partial r}U(t,r)Q(r)^{-1}x + U(t,r)\frac{d}{dr}Q(r)^{-1}x \\ &= -U(t,r)A(r)Q(r)^{-1}x - U(t,r)Q(r)^{-1}\dot{Q}(r)Q(r)^{-1}x \\ &= -\left(F(t,r)A(r) + F(t,r)\dot{Q}(r)Q(r)^{-1}\right)x \end{aligned} \tag{121}$$

where the last step uses the commutativity of an operator and its resolvent. Let $U_n(t,r) \to U(t,r)$ so that for any $y \in Y$

$$\frac{\partial}{\partial r}U_n(r,s)y = A_n(r)U_n(r,s)y. \tag{122}$$

Now,

$$\begin{aligned}\frac{\partial}{\partial r}U(t,r)Q(r)^{-1}U_n(r,s)y &= \frac{\partial}{\partial r}F(t,r)U_n(r,s)y \\ &= -F(t,r)\left(A(r) + \dot{Q}(r)Q(r)^{-1}\right)U_n(r,s)y \\ &\quad +F(t,r)A_n(r)U_n(r,s)y \\ &= -F(t,r)\left(A(r) + \dot{Q}(r)Q(r)^{-1} - A_n(r)\right)U_n(r,s)y \end{aligned} \tag{123}$$

The next step is to integrate the last expression from $s$ to $t$ with respect to $r$. This will be displayed.

$$\begin{aligned}
U(t,r)Q(r)^{-1}U_n(r,s)y|_s^t &= -\int_s^t F(t,r)\left(A(r)+\dot{Q}(r)Q(r)^{-1}-A_n(r)\right)U_n(r,s)y\,dr \\
Q(t)^{-1}U_n(t,s)y - U(t,s)Q(s)^{-1}y &= -\int_s^t U(t,r)Q(r)^{-1}\dot{Q}(r)Q(r)^{-1}U_n(r,s)y\,dr \\
&\quad + \int_s^t U(r,t)Q(r)^{-1}\left(A_n(r)-A(r)\right)U_n(r,s)y\,dr \qquad (124)
\end{aligned}$$

Note that the second integral vanishes in the limit. Then, from the denseness of $Y$ in $X$, it follows that

$$-Q(t)^{-1}U(t,s)x + U(t,s)Q(s)^{-1}x = \int_s^t U(t,r)Q(r)^{-1}\dot{Q}(r)Q(r)^{-1}U(r,s)x\,dr \qquad (125)$$

or

$$U(t,s)Q(s)^{-1}x = Q(t)^{-1}U(t,s)x + \int_s^t U(t,r)Q(r)^{-1}\dot{Q}(r)Q(r)^{-1}U(r,s)x\,dr. \qquad (126)$$

It is useful to compare this to the form of (119). Specifically, apply $Q^{-1}(t)$ to the left side of each operator in (119). Then it becomes

$$Q^{-1}(t)V(t,s)x = Q^{-1}(t)U(t,s)x + \int_s^t Q^{-1}(t)V(t,r)C(r)U(r,s)x\,dr. \qquad (127)$$

It is now clear that $Q^{-1}(t)V(t,s)x$, and $U(t,s)Q(s)^{-1}x$ are each solutions to the same integral equation, which is known to have a unique solution. Hence

$$U = Q^{-1}VQ \qquad (128)$$

as desired.

A construction of $V$ is now given. The purpose of this construction is to obtain a bound on $\|V\|_X$. Let

$$V_{(0)}(t,r)x = U(t,r)x \qquad (129)$$

It is easy to see that

$$\|V_{(0)}\|_X \leq M_{U_X}. \tag{130}$$

Next,

$$\begin{aligned}
V_{(1)}(t,r)x &= \int_r^t V_{(0)}(t,s)C(s)U(s,r)x\,ds \\
&= \int_r^t V_{(0)}(t,s)C(s)V_{(0)}(s,r)x\,ds \\
\|V_{(1)}(t,r)x\|_X &= \left\|\int_r^t V_{(0)}(t,s)C(s)V_{(0)}(s,r)x\,ds\right\|_X \\
&\leq \int_r^t \|V_{(0)}(t,s)C(s)V_{(0)}(s,r)x\|_X\,ds \\
&\leq \int_r^t \|V_{(0)}(t,s)\|_X\|C(s)V_{(0)}(s,r)x\|_X\,ds \\
&\leq M_{U_X}\int_r^t \|C(s)\|_X\|V_{(0)}(s,r)x\|_X\,ds \\
&\leq M_{U_X}\|C\|_\infty\int_r^t \|V_{(0)}(s,r)\|_X\|x\|_X\,ds \\
&\leq {M_{U_X}}^2\|C\|_\infty\|x\|_X\int_r^t ds \\
&\leq {M_{U_X}}^2\|C\|_\infty\|x\|_X(t-r) \\
&\leq {M_{U_X}}^2\|C\|_\infty\|x\|_X\hat{T}
\end{aligned} \tag{131}$$

Define

$$V_{(k+1)}(t,r)x = \int_r^t V_{(k)}(t,s)C(s)V_{(0)}(s,r)x\,ds. \tag{132}$$

Suppose now, for the purpose of proof by induction, that

$$\begin{aligned}
\left\|V_{(k)}(t,r)x\right\|_X &\leq {M_{U_X}}^{k+1}\|x\|_X\|C\|_\infty^k\frac{(t-r)^k}{k!} \\
&\leq {M_{U_X}}^{k+1}\|x\|_X\|C\|_\infty^k\frac{\hat{T}^k}{k!}.
\end{aligned} \tag{133}$$

The next step is to show that (133) holds for $V_{(k+1)}$. Then,

$$\begin{aligned}
\|V_{(k+1)}(t,r)x\|_X &= \left\| \int_r^t V_{(k)}(t,s)C(s)V_{(0)}(s,r)x\,ds \right\|_X \\
&\leq \int_r^t \|V_{(k)}(t,s)C(s)V_{(0)}(s,r)x\|_X\,ds \\
&\leq M_{U_X}{}^{k+1}\|C\|_\infty^k \int_r^t \|C(s)V_{(0)}(s,r)x\|_X \frac{(t-s)^k}{k!}ds \\
&= M_{U_X}{}^{k+2}\|C\|_\infty^{k+1}\|x\|_X \frac{(t-r)^{k+1}}{(k+1)!} \\
&\leq M_{U_X}{}^{k+2}\|C\|_\infty^{k+1}\|x\|_X \frac{\hat{T}^{k+1}}{(k+1)!} \qquad (134)
\end{aligned}$$

The operator $V$ obtained by the construction is

$$V(t,s) = \sum_{k=0}^{\infty} V_{(k)}(t,s). \qquad (135)$$

It is appropriate to verify that this expression for $V$ agrees with (119) of Lemma 43. The issue is whether $V$ satisfies (119) since (119) is known to have a unique solution. So, consider

$$\sum_{k=1}^{\infty} V_{(k)}(t,s)x = \int_s^t \sum_{k=0}^{\infty} V_{(k)}(t,r)C(r)V_{(0)}(r,s)x\,dr \qquad (136)$$

where the $k = 0$ term from the left hand side has cancelled with the first term on the right hand side of (119). Consider the first remaining summand on each side. By definition,

$$V_{(1)}(t,s)x = \int_s^t V_{(0)}(t,r)C(r)V_{(0)}(r,s)x\,dr. \qquad (137)$$

Likewise, for each succeeding pair of terms, the summand on the left equals the summand on the right. Hence, the sums are the same.

A bound for $V$ is available in terms of the bounds on the $V_{(k)}$. Namely,

$$\begin{aligned}
\|V\|_X &\leq \sum_{k=0}^{\infty} \|V_{(k)}\|_X \\
&\leq \sum_{k=0}^{\infty} M_{U_X}{}^{k+1}\,\|C\|_\infty^k \frac{\hat{T}^k}{k!}
\end{aligned}$$

$$\leq \; M_{U_X}\, e^{M_{U_X}\,\|C\|_\infty \hat{T}} \tag{138}$$

where the standard series representation for $e$ is easily recognized.

Everything is in place to bound $\|U\|_Y$. The superscripts will now be used again.

$$\begin{aligned} \|U^v\|_Y &= \|(Q^v)^{-1}V^vQ^v\|_Y \\ &\leq \|(Q^v)^{-1}\|_{X\to Y}\,\|V^v\|_X\,\|Q^v\|_{Y\to X} \\ &\leq \hat{\lambda}_Q\lambda_Q\|V^v\|_X \\ &\leq \hat{\lambda}_Q\lambda_Q M_{U_X}\, e^{M_{U_X}\,\|C^v\|_\infty \hat{T}} \end{aligned} \tag{139}$$

Recall that $C^v = \dot{Q}^v(Q^v)^{-1}$.

The final expression can be made more explicit in terms of fundamental quantities.

$$\begin{aligned} \|C^v(t)\|_\infty &= ess \sup_{t\in[0,\hat{T}]} \|\dot{Q}^v(t)Q^v(t)^{-1}\|_X \\ &\leq ess \sup_{t\in[0,\hat{T}]} \|\dot{Q}^v(t)\|_{Y\to X}\,\|Q^v(t)^{-1}\|_{X\to Y} \\ &\leq ess \sup_{t\in[0,\hat{T}]} \mu_Q(1+\hat{L})\hat{\lambda}_Q \\ &= \mu_Q(1+\hat{L})\hat{\lambda}_Q \end{aligned} \tag{140}$$

The combination of estimates (139) and (140) gives

$$\|U^v\|_Y \leq \hat{\lambda}_Q\lambda_Q \exp\left[(\lambda_N + 2\mu_N(1+\hat{L})\hat{T}) + \mu_Q(1+\hat{L})\hat{\lambda}_Q\hat{T}e^{(\lambda_N+2\mu_N(1+\hat{L})\hat{T})}\right]. \tag{141}$$

The bound given by the right hand side of this equation will be denoted by $M_{U_Y}$. This concludes the proof of the lemma. □

Thus, bounds on the evolution operators are obtained and the preliminaries for establishing a contraction mapping are complete. Note that the linear problem (108) is all that has been solved at this point. But, as it turns out, the solution is in $E$ and a mapping from $E \to E$ is established. Thenit will be established that the mapping is a contraction. The fixed point of the contraction map is the desired solution.

### 3.5.4 A contraction mapping

Attention now returns to the solution of (108) in terms of $U^v(t,s)$. In particular,

$$u^v(t) = U^v(t,0)u_0. \tag{142}$$

So, $u^v(t)$ is a continuous mapping of $[0,T]$ into $Y$ by properties 4 and 5 of Theorem 44. By property 2 it is differentiable for small $t$. Also, from the definition of a $Y$-valued solution, it is continuously differentiable into $X$.

It is desirable to establish that $u^v \in E$. This is the point in the argument where the choices of $\hat{L}$ and $\hat{T}$ occur. The conditions that $u^v$ must satisfy are:

$$\begin{aligned} \|u^v(t) - u_0\|_Y &\leq \hat{\rho} & (143)\\ \|u^v(t) - u^v(\hat{t})\|_X &\leq \hat{L}|t-\hat{t}|. & (144) \end{aligned}$$

It will now be shown that suitable choices are available to make this so.

**Lemma 60** *For each $v \in E$ and $t \in [0,\hat{T}]$, $u^v(t) \in E$.*

**Proof:** Consider the inequality (144) first. Note that $\frac{d}{dt}u^v(t) = -A^v(t)u^v(t)$. Recall that $M = \|u_0\|_Y$ and $R$ is the radius of a ball centered at $u_0$. Then,

$$\begin{aligned} \left\|u^v(t) - u^v(\hat{t})\right\|_X &\leq \sup_{t\in[0,\hat{T}]} \left\|\frac{d}{dt}u^v(t)\right\|_X |t-\hat{t}| \\ &= \|-A^v(t)u^v(t)\|_X|t-\hat{t}| \\ &\leq \|A^v(t)\|_{Y\to X}\ \|u^v(t)\|_Y|t-\hat{t}| \\ &\leq \lambda_A\|U^v(t,0)u_0\|_Y|t-\hat{t}| \\ &\leq \lambda_A\|U^v\|_Y\ \|u_0\|_Y|t-\hat{t}| \\ &\leq \lambda_A(R+M)\|U^v\|_Y\,|t-\hat{t}| \\ &\leq \lambda_A(R+M)\hat{\lambda}_Q\lambda_Q\exp\left[(\lambda_N + 2\mu_N(1+\hat{L})\hat{T})\right. \\ &\quad \left. +\mu_Q(1+\hat{L})\hat{\lambda}_Q\hat{T}e^{\lambda_N+2\mu_N(1+\hat{L})\hat{T}}\right]|t-\hat{t}|. \qquad (145) \end{aligned}$$

The goal is to have this quantity less than or equal to $\hat{L}|t - \hat{t}|$. But,

$$\hat{L} = 2\lambda_A(R + M)\hat{\lambda}_Q\lambda_Q e^{\lambda_N}\,. \tag{146}$$

Clearly $\hat{T}$ can be chosen small enough for the condition to be satisfied.

The inequality (143) is satisfied for some $\hat{T} > 0$ since $u^v(0) = u_0$ and $u^v$ is continuous, hence $u^v \in E$. □

Thus, a mapping $\Phi : E \to E$ is identified. In particular, for any $v \in E$, $\Phi(v)$ is obtained by the following steps:

1. Obtain an evolution operator $U^v(t,s)$ for the problem $\frac{d}{dt}u + A^v(t)u = 0$.
2. Let $u^v(t) = U^v(t,0)u_0$.
3. $\Phi(v) = u^v$.

The next item to establish is that $\Phi$ is in fact a contraction map. This will require a preliminary lemma.

**Lemma 61** *Let $U$ and $V$ be evolution operators corresponding to $A^u$ and $A^v$ respectively. Then*

$$V(t,r)y - U(t,r)y = \int_r^t U(t,s)(A^u(s) - A^v(s))V(s,r)y\,ds\,. \tag{147}$$

**Proof:** See Appendix D. □

The lemma will now be used to establish that $\Phi$ is a contraction mapping.

**Lemma 62** *The map $\Phi : E \to E$ is a contraction.*

**Proof:** The argument is straightforward. Let $u, v \in E$ be given. Let $\hat{u} = \Phi u$, $\hat{v} = \Phi v$.

$$\begin{aligned} d(\hat{u},\hat{v}) &= \sup_{0 \le t \le \hat{T}} \|\hat{u}(t) - \hat{v}(t)\|_X \\ &= \sup_{0 \le t \le \hat{T}} \|U^u(t,0)u_0 - U^v(t,0)u_0\|_X \end{aligned}$$

$$
\begin{aligned}
&= \sup_{0\le t\le \hat{T}} \left\| \int_0^t U^v(t,s)(A^u(s) - A^v(s))U^u(s,0)u_0\, ds \right\|_X \\
&\le \sup_{0\le t\le \hat{T}} \int_0^t \|U^v(t,s)(A^u(s) - A^v(s))U^u(s,0)u_0\|_X ds \\
&\le \sup_{0\le t\le \hat{T}} {M_{U_X}}^2 \mu_A d(u,v)\|u_0\|_X t \\
&\le {M_{U_X}}^2 \mu_A d(u,v)\|u_0\|_X \hat{T} \qquad (148)
\end{aligned}
$$

where $M_{U_X}$ is taken from Lemma 53.

It is now clear that $\hat{T}$ can be reduced if necessary to ensure that $\Phi$ is a contraction mapping. □

### *3.6 The proof of the theorem*

The Contraction Mapping Theorem, *eg* [54:pg 126], [13:pg 181], or [46:pp 40-42], applies to give the desired soluti. and the theorem follows as outlined at the beginning of Section 3.5. □

### *3.7 Applicability of Theorem 48 to an $\alpha(t)$ case*

Consider $u_{tt} + \alpha(t)u_{xxxx} = 0$ with the abstract formulation $u_t + A_7 u = 0;\ \ u(0) = u_0$ where

$$
A_7 = \begin{pmatrix} 0 & -1 \\ \alpha(t)D^4 & 0 \end{pmatrix}. \qquad (149)
$$

As before, $\alpha(t) \ge \alpha_{min} > 0$ for all $t \in [0,T]$. Furthermore, require $\alpha(t)$ to be continuously differentiable. Thus there is some $\alpha_{max}$ such that $\alpha(t) \le \alpha_{max}$ for all $t \in [0,T]$. Similarly there is some $\alpha'_{max}$ such that $\alpha'(t) \le \alpha'_{max}$ for all $t \in [0,T]$. For this equation, it has simply been assumed that $\alpha$ could vary with $t$ in a model originally derived for constant $\alpha$.

Notice that, in this particular application, the operator does not have any dependence on the solution. Thus the full power of the theorem is not exercised. The formal presence of $w$ in $A_7(t,w)$ will nevertheless be retained in the following exposition.

The space $X$ is almost the same as before. The point set is unchanged from (30) and (31). However, since the inner product on $X$ uses the value of $\alpha$ in its definition, it

would appear to have a variable norm. Though it will be useful to treat a family of norms corresponding to the various values of $\alpha$ this is not acceptable for the definition of the Banach space $X$. Since $\alpha$ is a continuous function of $t$ on a closed and bounded set, there is some $t_{min} \in [0, T]$ such that $\alpha$ attains its absolute minimum, say $\alpha_{min}$. The value $\alpha_{min}$ replaces $\alpha$ in the definition of the inner product on $X$ as given in Theorem 10. The space $Y$ is $D(A_3)$ with the norm defined as follows. The norm on $Y$ is a graph norm which depends on the operator $A$. Since the operator is now allowed to vary with $t$ and $w$ it is necessary to be careful so that the norm is well defined. To this end, choose $A_7(t_{min}, u_0)$ as the operator in the norm for $Y$,

$$\|y\|_Y = \|y\|_X + \|A_7(t_{min}, u_0)y\|_X. \tag{150}$$

The set $W$ is a ball in $Y$ with radius $R$ and center at the initial condition $u_0$. Other symbols with the same meaning as before are $M = \|u_0\|_Y$, and $Q = I + A$.

It is easily seen that Hypotheses 1-3, 5, and 8 of Theorem 48 are satisfied from the same arguments that were used in the proof of the theorem. For example, Hypothesis 8 is satisfied since $Q^{-1}$ is the resolvent of $A$ and resolvents commute with their generators. The other hypotheses must be addressed individually.

Consider Hypothesis 4. An upper bound on $\|Q(t, w)\|_{Y \to X}$ is desired. Let values of $t$ and $w$ be given. Note that $w$ does not have any role in this particular application. Then,

$$\begin{aligned}
\|Q(t,w)\|_{Y \to X} &= \|(I + A_7(t,w))\|_{Y \to X} \\
&= \sup_{0 \neq y \in Y} \frac{\|(I + A_7(t,w))y\|_X}{\|y\|_Y} \\
&= \sup_{0 \neq y \in Y} \frac{\|(I + A_7(t,w))y\|_X}{\|y\|_X + \|A_7(t_{min}, u_0)y\|_X} \\
&\leq \sup_{0 \neq y \in Y} \frac{\|y\|_X + \|A_7(t,w)y\|_X}{\|y\|_X + \|A_7(t_{min}, u_0)y\|_X}
\end{aligned}$$

$$= \sup_{0 \neq y \in Y} \frac{\|y\|_X + \left\| \begin{pmatrix} -y_2 \\ \alpha(t) D^4 y_1 \end{pmatrix} \right\|_X}{\|y\|_X + \|A_7(t_{min}, u_0) y\|_X}$$

$$= \sup_{0 \neq y \in Y} \frac{\|y\|_X + \left(\alpha_{min} \int_0^1 (y_2'')^2 \, dx + \alpha(t)^2 \int_0^1 (D^4 y_1)^2 \, dx\right)^{1/2}}{\|y\|_X + \|A_7(t_{min}, u_0) y\|_X} \tag{151}$$

The remainder of the argument is carried out in two separate cases. First, suppose that $\alpha_{min} \leq 1$. Then

$$\begin{aligned}
\|Q(t,w)\|_{Y \to X} &\leq \sup_{0 \neq y \in Y} \frac{\|y\|_X + \frac{\alpha(t)}{\alpha_{min}} \frac{\alpha_{min}}{\alpha(t)} \left(\alpha_{min} \int_0^1 (y_2'')^2 \, dx + \alpha(t)^2 \int_0^1 (D^4 y_1)^2 \, dx\right)^{1/2}}{\|y\|_X + \|A_7(t_{min}, u_0) y\|_X} \\
&= \sup_{0 \neq y \in Y} \frac{\|y\|_X + \frac{\alpha(t)}{\alpha_{min}} \left(\frac{\alpha_{min}^3}{\alpha(t)^2} \int_0^1 (y_2'')^2 \, dx + \alpha_{min}^2 \int_0^1 (D^4 y_1)^2 \, dx\right)^{1/2}}{\|y\|_X + \|A_7(t_{min}, u_0) y\|_X} \\
&\leq \sup_{0 \neq y \in Y} \frac{\|y\|_X + \frac{\alpha(t)}{\alpha_{min}} \left(\alpha_{min} \int_0^1 (y_2'')^2 \, dx + \alpha_{min}^2 \int_0^1 (D^4 y_1)^2 \, dx\right)^{1/2}}{\|y\|_X + \|A_7(t_{min}, u_0) y\|_X} \\
&\leq \sup_{0 \neq y \in Y} \frac{\|y\|_X + \frac{\alpha_{max}}{\alpha_{min}} \left(\alpha_{min} \int_0^1 (y_2'')^2 \, dx + \alpha_{min}^2 \int_0^1 (D^4 y_1)^2 \, dx\right)^{1/2}}{\|y\|_X + \|A_7(t_{min}, u_0) y\|_X} \\
&= \sup_{0 \neq y \in Y} \frac{\|y\|_X + \frac{\alpha_{max}}{\alpha_{min}} \|A_7(t_{min}, u_0) y\|_X}{\|y\|_X + \|A_7(t_{min}, u_0) y\|_X} \\
&\leq \frac{\alpha_{max}}{\alpha_{min}}.
\end{aligned} \tag{152}$$

For the second case, suppose $\alpha_{min} > 1$. Then

$$\begin{aligned}
\|Q(t,w)\|_{Y \to X} &\leq \sup_{0 \neq y \in Y} \frac{\|y\|_X + \left(\frac{\alpha(t)}{\alpha(t)}\right) \left(\alpha_{min} \int_0^1 (y_2'')^2 \, dx + \alpha(t)^2 \int_0^1 (D^4 y_1)^2 \, dx\right)^{1/2}}{\|y\|_X + \|A_7(t_{min}, u_0) y\|_X} \\
&= \sup_{0 \neq y \in Y} \frac{\|y\|_X + \alpha(t) \left(\frac{\alpha_{min}}{\alpha(t)^2} \int_0^1 (y_2'')^2 \, dx + \int_0^1 (D^4 y_1)^2 \, dx\right)^{1/2}}{\|y\|_X + \|A_7(t_{min}, u_0) y\|_X} \\
&\leq \sup_{0 \neq y \in Y} \frac{\|y\|_X + \alpha(t) \left(\alpha_{min} \int_0^1 (y_2'')^2 \, dx + \alpha_{min}^2 \int_0^1 (D^4 y_1)^2 \, dx\right)^{1/2}}{\|y\|_X + \|A_7(t_{min}, u_0) y\|_X} \\
&\leq \sup_{0 \neq y \in Y} \frac{\|y\|_X + \alpha_{max} \left(\alpha_{min} \int_0^1 (y_2'')^2 \, dx + \alpha_{min}^2 \int_0^1 (D^4 y_1)^2 \, dx\right)^{1/2}}{\|y\|_X + \|A_7(t_{min}, u_0) y\|_X}
\end{aligned}$$

$$\begin{aligned}
&\leq \alpha_{max} \sup_{0\neq y\in Y} \frac{\|y\|_X + \left(\alpha_{min}\int_0^1 (y_2'')^2\,dx + \alpha_{min}^2 \int_0^1 \left(D^4 y_1\right)^2\,dx\right)^{1/2}}{\|y\|_X + \|A_7(t_{min}, u_0)y\|_X} \\
&= \alpha_{max}. \qquad (153)
\end{aligned}$$

Therefore, choose $\lambda_Q = max\{\frac{\alpha_{max}}{\alpha_{min}}, \alpha_{max}\}$.

A lower bound on $\|Q(t,w)\|_{Y\to X}$ is useful to help identify an upper bound for $\|Q(t,w)^{-1}\|_{X\to Y}$. Since $-A_7(t,w)$ is dissipative,

$$\begin{aligned}
\|Q(t,w)y\|_X &= ([I + A_7(t,w)]y, [I + A_7(t,w)]y)^{1/2} \\
&= ((y,y) + 2(y, A_7(t,w)y) + (A_7(t,w), A_7(t,w)))^{1/2} \\
&\geq \left(\|y\|_X^2 + \|A_7(t,w)y\|_X^2\right)^{1/2} \\
&\geq 2^{-1/2}(\|y\|_X + \|A_7(t,w)y\|_X) \\
&= 2^{-1/2}\left(\|y\|_X + \left(\alpha_{min}\int_0^1 (y_2'')^2\,dx + \alpha(t)^2\int_0^1 \left(D^4 y_1\right)^2\,dx\right)^{1/2}\right) \\
&\geq 2^{-1/2}\left(\|y\|_X + \left(\alpha_{min}\int_0^1 (y_2'')^2\,dx + \alpha_{min}^2\int_0^1 \left(D^4 y_1\right)^2\,dx\right)^{1/2}\right) \\
&= 2^{-1/2}(\|y\|_X + \|A_7(t_{min}, w)\|_X) \\
&= 2^{-1/2}(\|y\|_X + \|A_7(t_{min}, u_0)\|_X) \\
&= 2^{-1/2}\|y\|_Y. \qquad (154)
\end{aligned}$$

It follows from Theorem 5.7.1 [54:pg 244] that an acceptable choice is $\hat{\lambda}_Q = 2^{1/2}$.

The next task is to calculate $\mu_Q$.

$$\begin{aligned}
\|Q(t,w) - Q(\hat{t},\hat{w})\|_{Y\to X} &= \sup_{0\neq y\in Y} \frac{\|\left(Q(t,w) - Q(\hat{t},\hat{w})\right)y\|_X}{\|y\|_Y} \\
&= \sup_{0\neq y\in Y} \frac{\|Q(t,w)y - Q(\hat{t},\hat{w})y\|_X}{\|y\|_Y} \\
&= \sup_{0\neq y\in Y} \frac{\|(I + A_7(t,w))y - \left(I + A_7(\hat{t},\hat{w})\right)y\|_X}{\|y\|_Y}
\end{aligned}$$

$$
\begin{aligned}
&= \sup_{0\neq y\in Y} \frac{\|A_7(t,w)y - A_7(\hat{t},\hat{w})y\|_X}{\|y\|_Y} \\
&= \sup_{0\neq y\in Y} \frac{\left\| \left(A_7(t,w) - A_7(\hat{t},\hat{w})\right) y \right\|_X}{\|y\|_Y} \\
&= \sup_{0\neq y\in Y} \frac{\left\| \begin{pmatrix} 0 & 0 \\ (\alpha(t)-\alpha(\hat{t}))D^4 & 0 \end{pmatrix} y \right\|_X}{\|y\|_Y} \\
&= \sup_{0\neq y\in Y} \frac{\left\| \begin{pmatrix} 0 \\ (\alpha(t)-\alpha(\hat{t}))D^4 y_1 \end{pmatrix} \right\|_X}{\|y\|_Y} \\
&\leq |\alpha(t)-\alpha(\hat{t})| \sup_{0\neq y\in Y} \frac{\left\| \begin{pmatrix} 0 \\ D^4 y_1 \end{pmatrix} \right\|_X}{\|y\|_Y} \\
&= |\alpha(t)-\alpha(\hat{t})| \sup_{0\neq y\in Y} \frac{\left\| \begin{pmatrix} 0 \\ D^4 y_1 \end{pmatrix} \right\|_X}{\|y\|_X + \|A_7(t_{min},\omega)y\|_X} \\
&= |\alpha(t)-\alpha(\hat{t})| \sup_{0\neq y\in Y} \frac{\left\| \begin{pmatrix} 0 \\ D^4 y_1 \end{pmatrix} \right\|_X}{\left\| \begin{pmatrix} y_1 \\ y_2 \end{pmatrix} \right\|_X + \left\| \begin{pmatrix} -y_2 \\ \alpha_{min} D^4 y_1 \end{pmatrix} \right\|_X} \\
&\leq |\alpha(t)-\alpha(\hat{t})| \sup_{0\neq y\in Y} \frac{\left\| \begin{pmatrix} 0 \\ D^4 y_1 \end{pmatrix} \right\|_X}{\left\| \begin{pmatrix} 0 \\ \alpha_{min} D^4 y_1 \end{pmatrix} \right\|_X}
\end{aligned}
$$

$$= \quad |\alpha(t) - \alpha(\hat{t})| \frac{1}{\alpha_{min}} \sup_{0 \neq y \in Y} \frac{\left\| \begin{pmatrix} 0 \\ D^4 y_1 \end{pmatrix} \right\|_X}{\left\| \begin{pmatrix} 0 \\ D^4 y_1 \end{pmatrix} \right\|_X}$$

$$= \quad \alpha'_{max} |t - \hat{t}| \frac{1}{\alpha_{min}}$$

$$= \quad \frac{\alpha'_{max}}{\alpha_{min}} |t - \hat{t}| \tag{155}$$

So, choose $\mu_Q = \frac{\alpha'_{max}}{\alpha_{min}}$. This completes the arguments to satisfy Hypothesis 5. Notice that in this application the isomorphism does not depend on $w$.

Consider Hypothesis 6. Let $N(t,w)y = \alpha(t) \int_0^1 (y_1'')^2 \, dx + \int_0^1 y_2{}^2 \, dx$. Then

$$d(N(t,w), \|\cdot\|_X) = log \sup_{0 \neq y \in X} max \left\{ \frac{\|y\|_t}{\|y\|_X}, \frac{\|y\|_X}{\|y\|_t} \right\}$$

$$= \quad log \sup_{0 \neq y \in X} max \left\{ \frac{\alpha(t) \int_0^1 (y_1'')^2 \, dx + \int_0^1 y_2{}^2 \, dx}{\alpha_{min} \int_0^1 (y_1'')^2 \, dx + \int_0^1 y_2{}^2 \, dx}, \frac{\alpha_{min} \int_0^1 (y_1'')^2 \, dx + \int_0^1 y_2{}^2 \, dx}{\alpha(t) \int_0^1 (y_1'')^2 \, dx + \int_0^1 y_2{}^2 \, dx} \right\}$$

$$\leq \quad log \sup_{0 \neq y \in X} \left\{ \frac{\alpha(t)}{\alpha_{min}} \right\}$$

$$\leq \quad log \frac{\alpha_{max}}{\alpha_{min}}. \tag{156}$$

So, choose $\lambda_N = log \frac{\alpha_{max}}{\alpha_{min}}$. It is easy to see that $\mu_N = \lambda_N$ is suitable for the current problem.

The next argument requires a preliminary lemma. The lemma is elementary but is included to help clarify the argument.

**Lemma 63** *For positive real numbers $a, b,$ and $c$ with $b \geq c$, it follows that $\frac{a+b}{a+c} \leq \frac{b}{c}$.*

**Proof:** See Appendix D. □

Consider Hypothesis 7. Claim: $A_7(t,w) \in G(X_{N(t,w)}, 1, 0)$. Recall that $X_{N(t,w)}$ is simply the set $X$ with the norm that uses $\alpha(t)$. For each $(t,w)$ this reduces to the constant coefficient case. In particular, for each fixed value of $t$, Lemmas 21 and 26 and Theorem 27 apply to $A_7(t,w)$ just as they did to $A_1$.

Claim: $\|A_7(t,w)\|_{Y\to X} \le \lambda_A$. This is straightforward, as follows.

$$\begin{aligned}
\|A_7(t,w)\|_{Y\to X} &= \sup_{0\neq y\in Y} \frac{\|A_7(t,w)y\|_X}{\|y\|_Y} \\
&= \sup_{0\neq y\in Y} \frac{\|A_7(t,w)y\|_X}{\|y\|_X + \|A_7(t_{min},w)y\|_X} \\
&\le \sup_{0\neq y\in Y} \frac{\|A_7(t,w)y\|_X}{\|A_7(t_{min},w)y\|_X} \\
&= \sup_{0\neq y\in Y} \frac{\left\| \begin{pmatrix} -y_2 \\ \alpha(t)D^4 y_1 \end{pmatrix} \right\|_X}{\left\| \begin{pmatrix} -y_2 \\ \alpha_{min} D^4 y_1 \end{pmatrix} \right\|_X} \\
&= \sup_{0\neq y\in Y} \frac{\left(\alpha_{min}\int_0^1 (y_2'')^2\,dx + \alpha(t)\int_0^1 (D^4 y_1)^2\,dx\right)^{1/2}}{\left(\alpha_{min}\int_0^1 (y_2'')^2\,dx + \alpha_{min}\int_0^1 (D^4 y_1)^2\,dx\right)^{1/2}} \\
&\le \sup_{0\neq y\in Y} \frac{\left(\alpha_{min}\int_0^1 (y_2'')^2\,dx + \alpha_{max}\int_0^1 (D^4 y_1)^2\,dx\right)^{1/2}}{\left(\alpha_{min}\int_0^1 (y_2'')^2\,dx + \alpha_{min}\int_0^1 (D^4 y_1)^2\,dx\right)^{1/2}} \\
&\le \frac{\alpha_{max}}{\alpha_{min}} \qquad (157)
\end{aligned}$$

Choose $\lambda_A = \frac{\alpha_{max}}{\alpha_{min}}$.

It is suitable to choose $\mu_A = \mu_Q$. Continuity holds as argued in the $\beta(t)$ case.

This completes the verification of hypotheses for this case. The theorem applies and guarantees the existence of a unique solution.

### 3.8 Chapter summary

A theorem for existence and uniqueness of solutions to a broad class of abstract Cauchy problems has been presented, along with its proof. Also, an application has been described, formulated, and shown to satisfy the hypotheses. The existence of a unique solution is guaranteed.

## IV. A nonlinear damping term

Consider $u_{tt} + D^2(\beta(u)D^2u_t) + \alpha D^4 u = 0$. Formulate this as an abstract system as before with

$$A = A(u) = \begin{pmatrix} 0 & -1 \\ \alpha D^4 & D^2(\beta(u)D^2) \end{pmatrix} \tag{158}$$

where, in the abstract system, $u$ is a vector with components $u_1$ and $u_2$. The space $X$ and $D(A)$ are the same as described in the previous chapter.

It is convenient and informative to treat a specific case and demonstrate how the conditions are verified.

Consider $u_{tt} + D^2((\beta_0 + \beta_1 u)D^2u_t) + \alpha D^4 u = 0$ where $\alpha$, $\beta_0$, and $\beta_1$ are constants. Formulate this as an abstract system as before with

$$A_8 = A(u) = \begin{pmatrix} 0 & -1 \\ \alpha D^4 & D^2((\beta_0 + \beta_1 u_1)D^2) \end{pmatrix} \tag{159}$$

and $D(A_8) = D(A_3)$. Certain restrictions will be placed on $\beta_0$ in terms of other constants in the problem. Portions of the problem formulation which are different from before are described next.

### 4.1 Preparation for application of the existence theorem

Let $Y$ be the set of points (pairs of functions) in $D(A_8)$ endowed with the norm

$$\|y\|_Y = \|y\|_X + \|D^4 y_1\|_{L_2} + \|D^4 y_2\|_{L_2}. \tag{160}$$

Notice that $D(A_8)$ does not depend on $t$. The linear space $Y$ is complete as shown before. The graph norm has not been used this time. Use of the graph norm, according to the previous pattern, would require $W$ to be closed and bounded. The current strategy uses the norm (160).

Some earlier terminology is now reviewed. Let $W = B(u_0; R)$ be an open ball in $Y$ with center $u_0$ (the initial condition vector) and radius $R$ (not yet specified). Let $M = \|u_0\|_Y$.

Several lemmas are now presented. They will be used to identify the restrictions on $\beta_0$ and in other computations. The purpose of these lemmas is to identify bounds on $y_2''$ and $y_2'''$ in terms of $D^4 y_2$.

**Lemma 64** *For any* $c \in [0,1]$ and $y_2 \in H^3$, $\|y_2''\|_\infty \leq |y_2''(c)| + \|y_2'''\|_{L_2}$.

**Proof:** This follows immediately from the same line of argument as in the proof of Lemma 12. □

**Lemma 65** *For any* $y_2 \in H^3$ *with* $y_2'(0) = y_2'(1) = 0$ *there is some* $c \in [0,1]$, *such that* $y_2''(c) = 0$.

**Proof:** Note the continuity of $y_2''$ as an element of $H^3$ and the boundary conditions $y_2'(0) = y_2'(1) = 0$. The existence of the desired $c \in [0,1]$, such that $y_2''(c) = 0$ is immediate from Rolle's Theorem. □

It is useful to note that there are at least two such values, say $c_1$ and $c_2$. This is because an application of Rolle's Theorem to the continuous function $y_2$ and the boundary conditions on $y_2$ gives a point $\hat{c} \in (0,1)$ such that $y_2'(\hat{c}) = 0$. Then the argument of Lemma 65 can be applied to each of the intervals $(0, \hat{c})$ and $(\hat{c}, 1)$.

**Corollary 66** *For* $y_2 \in H^3$ *and satisfying the boundary conditions,*

$$\|y_2''\|_\infty \leq \|y_2'''\|_{L_2}. \tag{161}$$

**Lemma 67** *For any* $c \in [0,1]$ *and* $y_2 \in H^4$,

$$\|y_2'''\|_\infty \leq |y_2'''(c)| + \|D^4 y_2\|_{L_2}. \tag{162}$$

**Proof:** This follows immediately from the same line of argument as in the proof of Lemma 12. □

But, again by Rolle's theorem, there is some $\hat{c} \in [c_1, c_2]$ such that $y_2'''(\hat{c}) = 0$. Thus, the following corollary is immediate.

**Corollary 68** *For $y_2 \in H^4$ and satisfying the boundary conditions,*

$$\|y_2'''\|_\infty \leq \|D^4 y_2\|_{L_2}. \tag{163}$$

**Lemma 69** *For $w \in B(u_0; R) = W$, $|w_1| < \frac{(M+R)}{\alpha^{1/2}}$ where $M = \|u_0\|_Y$ and $\alpha$ is the constant in (159) which is also used in the definition of $\|\cdot\|_X$.*

Proof: Since $w \in W$, it is clear that $\|w\|_Y < R + M$. Recall that components of elements of $Y$ satisfy the boundary conditions in the specification of $D(A)$. From the definitions it is not difficult to see that

$$\begin{aligned} |w_1| &\leq \|w_1\|_\infty \\ &\leq \|w_1''\|_{L_2} \\ &= \frac{1}{\alpha^{1/2}} \left\| \begin{pmatrix} w_1 \\ 0 \end{pmatrix} \right\|_X \\ &\leq \frac{1}{\alpha^{1/2}} \|w\|_X \\ &\leq \frac{1}{\alpha^{1/2}} \|w\|_Y \\ &\leq \frac{M+R}{\alpha^{1/2}}. \quad \square \end{aligned} \tag{164}$$

The restriction that $\beta_0$ will be required to satisfy is now given. It is required that, for every $x \in [0,1]$, $\beta_0 + \beta_1 w_1 \geq 0$ for all $w \in W$. Since $\|w_1\|_\infty \leq \frac{R+M}{\alpha^{1/2}}$ this can be satisfied with a finite choice for $\beta_0$. Or, more to the point, it is desirable that $\frac{\beta_0}{\beta_1} \geq -w_1(x)$ for all $x \in [0,1]$. Note that the choice of small values for $R$ and $M$ will allow more flexibility in the choice of acceptable $\beta_0$ and $\beta_1$.

### *4.2 Applying Theorem 48*

Consider whether the nonlinear damping problem satisfies the hypotheses of Theorem 48. Hypotheses 1 and 2 are clearly satisfied. Hypothesis 3 is satisfied with $Q = I + A$ as before.

Bounds on $Q$ and $Q^{-1}$ for Hypothesis 4 may be obtained with some effort. In order to obtain such bounds most easily, several elementary lemmas are presented.

**Lemma 70** *For* $a, b \geq 0$, $(a^2 + b^2)^{1/2} \leq a + b$.

**Proof:** See Appendix D. □

**Lemma 71** *For any real numbers* $a$ *and* $b$, $(a + b)^2 \leq 2(a^2 + b^2)$.

**Proof:** See Appendix D. □

**Corollary 72** *For positive real numbers* $a$ *and* $b$, $2^{-1/2}(a^{1/2} + b^{1/2}) \leq (a + b)^{1/2}$.

**Lemma 73** *For positive real numbers* $a, b$, *and* $c$ *it holds that* $(a+b+c)^2 \leq 3(a^2+b^2+c^2)$.

**Proof:** See Appendix D. □

**Lemma 74** *For any positive, real* $a, b$, *and* $c$, *it holds that* $(a+b+c)^{1/2} \leq (a^{1/2}+b^{1/2}+c^{1/2})$.

**Proof:** See Appendix D. □

Consider a bound for $Q$. Equation (40) will be used several times in the following string of inequalities.

$$\|Q(w)\|_{Y \to X} = \|I + A_8(w)\|_{Y \to X}$$

$$= \sup_{0 \neq y \in Y} \frac{\|(I + A_8(w))y\|_X}{\|y\|_Y}$$

$$= \sup_{0 \neq y \in Y} \frac{\|y + A_8(w)y\|_X}{\|y\|_X + \|D^4 y_1\|_{L_2} + \|D^4 y_2\|_{L_2}}$$

$$\leq \sup_{0\neq y\in Y} \frac{\|y\|_X + \|A_8(w)y\|_X}{\|y\|_X + \|D^4y_1\|_{L_2} + \|D^4y_2\|_{L_2}}$$

$$\leq \sup_{0\neq y\in Y} \frac{\|y\|_X}{\|y\|_X + \|D^4y_1\|_{L_2} + \|D^4y_2\|_{L_2}} + \sup_{0\neq y\in Y} \frac{\|A_8(w)y\|_X}{\|y\|_X + \|D^4y_1\|_{L_2} + \|D^4y_2\|_{L_2}}$$

$$\leq 1 + \sup_{0\neq y\in Y} \frac{\|A_8(w)y\|_X}{\|y\|_X + \|D^4y_1\|_{L_2} + \|D^4y_2\|_{L_2}}$$

$$\leq 1 + \sup_{0\neq y\in Y} \frac{\|A_8(w)y\|_X}{\|D^4y_1\|_{L_2} + \|D^4y_2\|_{L_2}}$$

$$= 1 + \sup_{0\neq y\in Y} \frac{\left(\alpha \int_0^1 (y_2'')^2\, dx + \int_0^1 \left(\alpha D^4y_1 + \beta_0 D^4y_2 + \beta_1 D^2\left(w_1 D^2 y_2\right)\right)^2 dx\right)^{1/2}}{\|D^4y_1\|_{L_2} + \|D^4y_2\|_{L_2}}$$

$$\leq 1 + \sup_{0\neq y\in Y} \frac{\alpha^{1/2}\left(\int_0^1 (y_2'')^2\, dx\right)^{1/2}}{\|D^4y_1\|_{L_2} + \|D^4y_2\|_{L_2}}$$

$$+ \sup_{0\neq y\in Y} \frac{\left(\int_0^1 \left(\alpha D^4y_1 + \beta_0 D^4y_2 + \beta_1 D^2\left(w_1 D^2 y_2\right)\right)^2 dx\right)^{1/2}}{\|D^4y_1\|_{L_2} + \|D^4y_2\|_{L_2}}$$

$$\leq 1 + \alpha^{1/2} + \sup_{0\neq y\in Y} \frac{\left(\int_0^1 \left(\alpha D^4y_1 + \beta_0 D^4y_2 + \beta_1 D^2\left(w_1 D^2 y_2\right)\right)^2 dx\right)^{1/2}}{\|D^4y_1\|_{L_2} + \|D^4y_2\|_{L_2}}$$

$$\leq 1 + \alpha^{1/2} + 2^{1/2} \sup_{0\neq y\in Y} \frac{\left(\int_0^1 \left(\alpha D^4y_1\right)^2 dx + \int_0^1 \left(\beta_0 D^4y_2 + \beta_1 D^2\left(w_1 D^2 y_2\right)\right)^2 dx\right)^{1/2}}{\|D^4y_1\|_{L_2} + \|D^4y_2\|_{L_2}}$$

$$\leq 1 + \alpha^{1/2} + 2^{1/2} \sup_{0\neq y\in Y} \frac{\left(\int_0^1 \left(\alpha D^4y_1\right)^2 dx\right)^{1/2} + \left(\int_0^1 \left(\beta_0 D^4y_2 + \beta_1 D^2\left(w_1 D^2 y_2\right)\right)^2 dx\right)^{1/2}}{\|D^4y_1\|_{L_2} + \|D^4y_2\|_{L_2}}$$

$$\leq 1 + \alpha^{1/2} + 2^{1/2}\alpha + 2^{1/2} \sup_{0\neq y\in Y} \frac{2^{1/2}\left(\int_0^1 \left(\beta_0 D^4y_2\right)^2 dx + \int_0^1 \left(\beta_1 D^2\left(w_1 D^2 y_2\right)\right)^2 dx\right)^{1/2}}{\|D^4y_1\|_{L_2} + \|D^4y_2\|_{L_2}}$$

$$\leq 1 + \alpha^{1/2} + 2^{1/2}\alpha + 2 \sup_{0\neq y\in Y} \frac{\|\beta_0 D^4y_2\|_{L_2}}{\|D^4y_1\|_{L_2} + \|D^4y_2\|_{L_2}}$$

$$+2 \sup_{0 \neq y \in Y} \frac{\left(\int_0^1 \left(\beta_1 D^2 \left(w_1 D^2 y_2\right)\right)^2 \, dx\right)^{1/2}}{\|D^4 y_1\|_{L_2} + \|D^4 y_2\|_{L_2}}$$

$$= \quad 1 + \alpha^{1/2} + 2^{1/2}\alpha + 2|\beta_0| + 2 \sup_{0 \neq y \in Y} \frac{|\beta_1| \left(\int_0^1 \left(w_1 D^4 y_2 + 2w_1' D^3 y_2 + w_1'' D^2 y_2\right)^2 \, dx\right)^{1/2}}{\|D^4 y_1\|_{L_2} + \|D^4 y_2\|_{L_2}}$$

$$\leq \quad 1 + \alpha^{1/2} + 2^{1/2}\alpha + 2|\beta_0|$$
$$+2 \sup_{0 \neq y \in Y} \frac{3^{1/2}|\beta_1| \left(\int_0^1 {w_1}^2 (D^4 y_2)^2 \, dx + 2\int_0^1 (w_1')^2 (D^3 y_2)^2 \, dx + \int_0^1 (w_1'')^2 (D^2 y_2)^2 \, dx\right)^{1/2}}{\|D^4 y_1\|_{L_2} + \|D^4 y_2\|_{L_2}}$$

$$\leq \quad 1 + \alpha^{1/2} + 2^{1/2}\alpha + 2|\beta_0|$$
$$+2 \cdot 3^{1/2}|\beta_1| \sup_{0 \neq y \in Y} \frac{(R + M) \left(\int_0^1 (D^4 y_2)^2 \, dx + 2\int_0^1 (D^3 y_2)^2 \, dx + \int_0^1 (D^2 y_2)^2 \, dx\right)^{1/2}}{\|D^4 y_1\|_{L_2} + \|D^4 y_2\|_{L_2}}$$

$$\leq \quad 1 + \alpha^{1/2} + 2^{1/2}\alpha + 2|\beta_0|$$
$$+2 \cdot 3^{1/2}|\beta_1|(R + M) \sup_{0 \neq y \in Y} \frac{\|D^4 y_2\|_{L_2} + 2\|D^3 y_2\|_{L_2} + \|D^2 y_2\|_{L_2}}{\|D^4 y_1\|_{L_2} + \|D^4 y_2\|_{L_2}}$$

$$\leq \quad 1 + \alpha^{1/2} + 2^{1/2}\alpha + 2|\beta_0| + 8 \cdot 3^{1/2}|\beta_1|(R + M) \tag{165}$$

So, choose $\lambda_Q = 1 + \alpha^{1/2} + 2^{1/2}\alpha + 2|\beta_0| + 8 \cdot 3^{1/2}|\beta_1|(R + M)$.

Now that $Q$ is bounded, recall that it is also linear, one-to-one, and onto. Then, by a standard corollary (see [22:pg 47], or [73:pg 77]) of the Open Mapping Theorem, $Q^{-1}$ is also bounded, say by $\hat{\lambda}_Q$.

A suitable choice for $\mu_Q$ is $\mu_A$ which is determined in (168).

Hypotheses 5 and 6 are trivially satisfied since the variable norms are not used in this application.

The first part of Hypothesis 7 is satisfied for each $w \in W$ as argued in the $\beta(x)$ case. For clarity, the dissipative argument, which appears on the surface to be different, is

presented. First, some notation is clarified.

$$
\begin{aligned}
A_8 &= \begin{pmatrix} 0 & -1 \\ \alpha D^4 & D^2((\beta_0+\beta_1 u_1)D^2) \end{pmatrix} \\
A_8(u)u &= \begin{pmatrix} 0 & -1 \\ \alpha D^4 & D^2((\beta_0+\beta_1 u_1)D^2) \end{pmatrix} \begin{pmatrix} u_1 \\ u_2 \end{pmatrix} \\
&= \begin{pmatrix} -u_2 \\ \alpha D^4 u_1 + D^2\left((\beta_0+\beta_1 u_1)D^2 u_2\right) \end{pmatrix} \\
A_8(w)u &= \begin{pmatrix} -u_2 \\ \alpha D^4 u_1 + D^2\left((\beta_0+\beta_1 w_1)D^2 u_2\right) \end{pmatrix} \\
(A_8(w)u,u) &= -\alpha\int_0^1 u_2'' u_1''\,dx + \int_0^1 u_2 \alpha D^4 u_1\,dx + \int_0^1 u_2 D^2\left((\beta_0+\beta_1 w_1)D^2 u_2\right)dx \\
&= u_2 D\left((\beta_0+\beta_1 w_1)D^2 u_2\right)\Big|_0^1 - \int_0^1 Du_2 D\left((\beta_0+\beta_1 w_1)D^2 u_2\right)dx \\
&= -Du_2\left((\beta_0+\beta_1 w_1)D^2 u_2\right)\Big|_0^1 + \int_0^1 D^2 u_2(\beta_0+\beta_1 w_1)D^2 u_2\,dx \\
&= \int_0^1 (\beta_0+\beta_1 w_1)(D^2 u_2)^2\,dx \qquad (166)
\end{aligned}
$$

where, in the line with three integrals, the first two cancel out after integration by parts. Also, the boundary terms, arising from integrations by parts, are all zero. If it happens that the quantity $\beta_0+\beta_1 w_1$ is greater than or equal to zero for all $x$ (which is required above) then $-A_8(w)$ is dissipative. The remainder of the argument to establish $A_8(w) \in G(X,1,0)$ is the same as for the $\beta(x)$ case. This completes the discussion of the first part of Hypothesis 7.

For the second part of Hypothesis 7 note that, from the work for $\lambda_Q$ leading to (165), it follows that

$$\lambda_A = \alpha^{1/2} + 2^{1/2}\alpha + 2|\beta_0| + 8\cdot 3^{1/2}|\beta_1|(R+M) \qquad (167)$$

is suitable.

Next, a suitable $\mu_A$ must be identified for the third part of Hypothesis 7. Notice that in the current application there is no explicit time dependence in $A_8$. For the first inequality, in the following sequence of equalities and inequalities, a comment is appropriate. The first two terms in the integrand are bounded in terms of the supremum norm on the factor involving $w_1$. The third term in the integrand is bounded in terms of the supremum norm on the $D^2y_2$ factor which leads to an $L_2$ bound on $D^4y_2$. Now, by definition

$$\|A_8(w) - A_8(\hat{w})\|_{Y\to X} = \sup_{0\neq y\in Y} \frac{\|(A_8(w) - A_8(\hat{w}))y\|_X}{\|y\|_Y}$$

$$= \sup_{0\neq y\in Y} \frac{\|(A_8(w) - A_8(\hat{w}))y\|_X}{\|y\|_X + \|D^4y_1\|_{L_2} + \|D^4y_2\|_{L_2}}$$

$$= \sup_{0\neq y\in Y} \frac{\left\| \begin{pmatrix} 0 & 0 \\ 0 & \beta_1 D^2\left((w_1 - \hat{w}_1)D^2\right) \end{pmatrix} \begin{pmatrix} y_1 \\ y_2 \end{pmatrix} \right\|_X}{\|y\|_X + \|D^4y_1\|_{L_2} + \|D^4y_2\|_{L_2}}$$

$$= \sup_{0\neq y\in Y} \frac{\left\| \begin{pmatrix} 0 \\ \beta_1 D^2\left((w_1 - \hat{w}_1)D^2y_2\right) \end{pmatrix} \right\|_X}{\|y\|_X + \|D^4y_1\|_{L_2} + \|D^4y_2\|_{L_2}}$$

$$= \sup_{0\neq y\in Y} \frac{\beta_1 \left(\int_0^1 \left(D^2\left((w_1 - \hat{w}_1)D^2y_2\right)\right)^2 dx\right)^{1/2}}{\|y\|_X + \|D^4y_1\|_{L_2} + \|D^4y_2\|_{L_2}}$$

$$= |\beta_1| \sup_{0\neq y\in Y} \frac{\left(\int_0^1 \left((w_1 - \hat{w}_1)D^4y_2 + 2(w_1 - \hat{w}_1)'D^3y_2 + (w_1 - \hat{w}_1)''D^2y_2\right)^2 dx\right)^{1/2}}{\|y\|_X + \|D^4y_1\|_{L_2} + \|D^4y_2\|_{L_2}}$$

$$\leq |\beta_1| \sup_{0\neq y\in Y} \frac{\left(3\int_0^1 ((w_1 - \hat{w}_1)^2(D^4y_2)^2 + 4({w_1}' - {\hat{w}_1}')^2(D^3y_2)^2 + ({w_1}'' - {\hat{w}_1}'')^2(D^2y_2)^2)dx\right)^{1/2}}{\|D^4y_2\|_{L_2}}$$

$$\leq 3^{1/2}|\beta_1| \sup_{0\neq y\in Y} \frac{\left(\left(\|({w_1}'' - {\hat{w}_1}'')\|_{L_2}^2 + 4\|({w_1}'' - {\hat{w}_1}'')\|_{L_2}^2 + \|{w_1}'' - {\hat{w}_1}''\|_{L_2}^2\right) \|D^4y_2\|_{L_2}^2\right)^{1/2}}{\|D^4y_2\|_{L_2}}$$

$$= 3^{1/2}|\beta_1| 6^{1/2} \|{w_1}'' - {\hat{w}_1}''\|_{L_2}$$

$$= 3 \cdot 2^{1/2} |\beta_1| \, \|w - \hat{w}\|_X \tag{168}$$

so it is acceptable to choose $\mu_A = 3 \cdot 2^{1/2}|\beta_1|$.

Hypothesis 8 follows immediately, as before, since $Q^{-1}$ is the resolvent of $A_8$.

Identification of an acceptable $\hat{T}$ is a several step process. Since this process has been demonstrated in detail for a previous application, it will not be repeated here.

The first step of the solution algorithm is now described. Apply the arguments of the $\beta(x)$ case to $A_8(u_0)$. This yields a solution $\hat{u}(t)$ based on the single operator $A_8(u_0)$. Next apply the iterative scheme, as in the $\beta(t, x)$ case. to the operator family $A_8(\hat{u}(t))$. This gives an evolution system $\hat{U}(t, s)$ which generates a solution $\hat{u}(t)$. This is just one step of the iteration. The fixed point algorithm (from the proof of Theorem 48) guarantees that the iterates will converge.

The continuous dependence result, Theorem 3.2 of [35:pp 170-171], is applicable to this problem.

### 4.3 A numerical example

In this section a specific example is presented. The equation is the same as earlier in the chapter, but now the constants take on specific values. Also, a specific initial condition vector is given. The example and the software to propagate its solution are from [60].

Choose the following values for the constants.

$$\begin{aligned} a &= .008 \\ \beta_0 &= .01 \\ \beta_1 &= .001 \end{aligned} \tag{169}$$

For the first component of $u_0$ choose

$$u_1 = \sin \nu x + \frac{\sinh \nu - \sin \nu}{\cos \nu - \cosh \nu} \cos \nu x - \frac{\sinh \nu - \sin \nu}{\cos \nu - \cosh \nu} \cosh \nu - \sinh \nu. \tag{170}$$

This is shown in Figure 4 with $\nu = 4.73$. The choice of $\nu$ satisfies $cos(\nu)cosh(\nu) = 1$. This is required for $u_0$ to satisfy the boundary conditions necessary for it to be in $D(A_8)$.

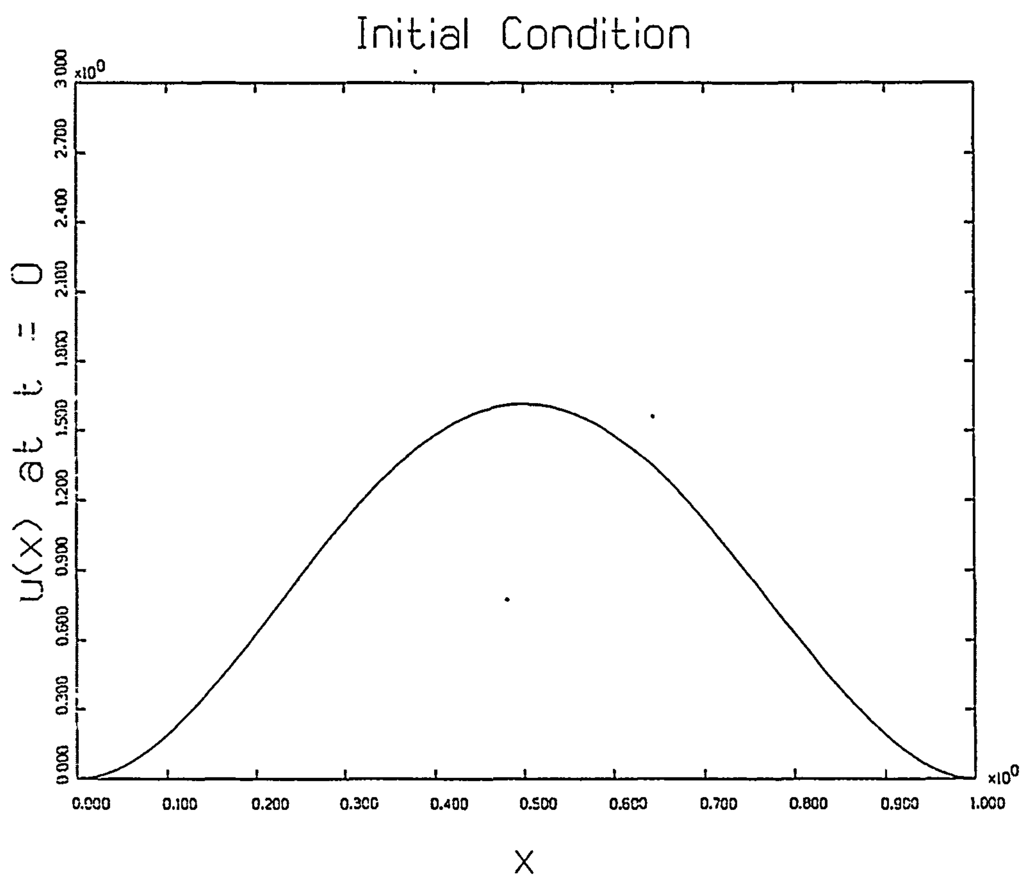


Figure 4. Initial value curve for the example

Choose 0 for the second component of $u_0$.

If $\beta_1$ were zero, then this problem could be solved by separation of variables. The fundamental frequency corresponds, in this case, to $\nu = 4.73$ approximately. Choose this value for $\nu$ and test the basic algorithm for convergence.

The solution algorithm is outlined now.

1. Establish values for the constants in the problem.
2. Establish a grid and assign an initial estimate to the solution at each grid point. As the first estimate suppose that the initial condition vector is a constant solution.
3. Initialize a counter for time increments, say $j = 0$.
4. Evaluate the quantity $\beta_0 + \beta_1 u$ for each grid point, based on the current estimate for $u$.
5. Increment the counter: $j = j + 1$.
6. Propogate the approximate solution from $t = (j-1)\Delta t$ to $t = j\Delta t$. (This step is done with the program DGEAR from the standard Fortran package known as IMSL.)
7. If the desired final time has not been reached, go to step 5.
8. If the solution has not converged, go to step 3.
9. This completes the algorithm.

The results are presented in Figure 5. The plot is for the vertical displacement of the midpoint of the beam. The line across the top represents the initial estimate, which is the initial condition as a constant solution. The next curve down is the estimate after one iteration of the algorithm. The third curve is the estimate after two iterations. The third iteration lies on top of the second one and cannot be distinguished, though it has been included.

A second example, with $\beta_1 = .008$, is shown in Figure 6. Notice that while the spacing is different, the basic character is the same.

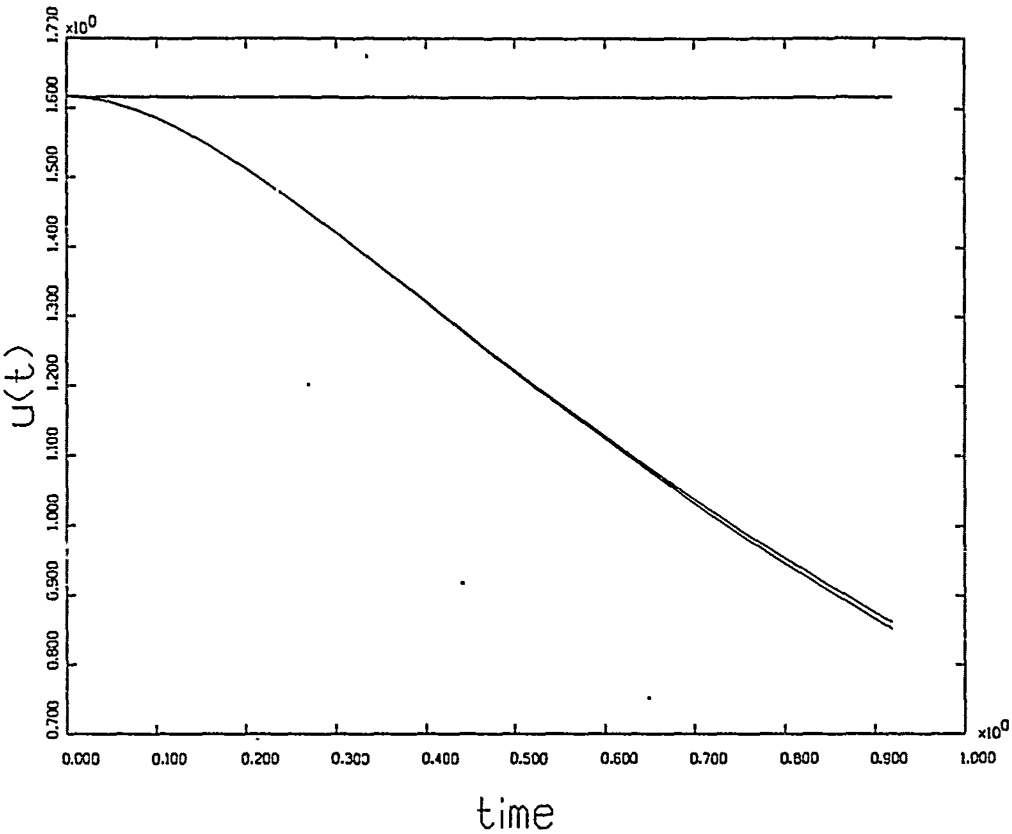


Figure 5. Example 1

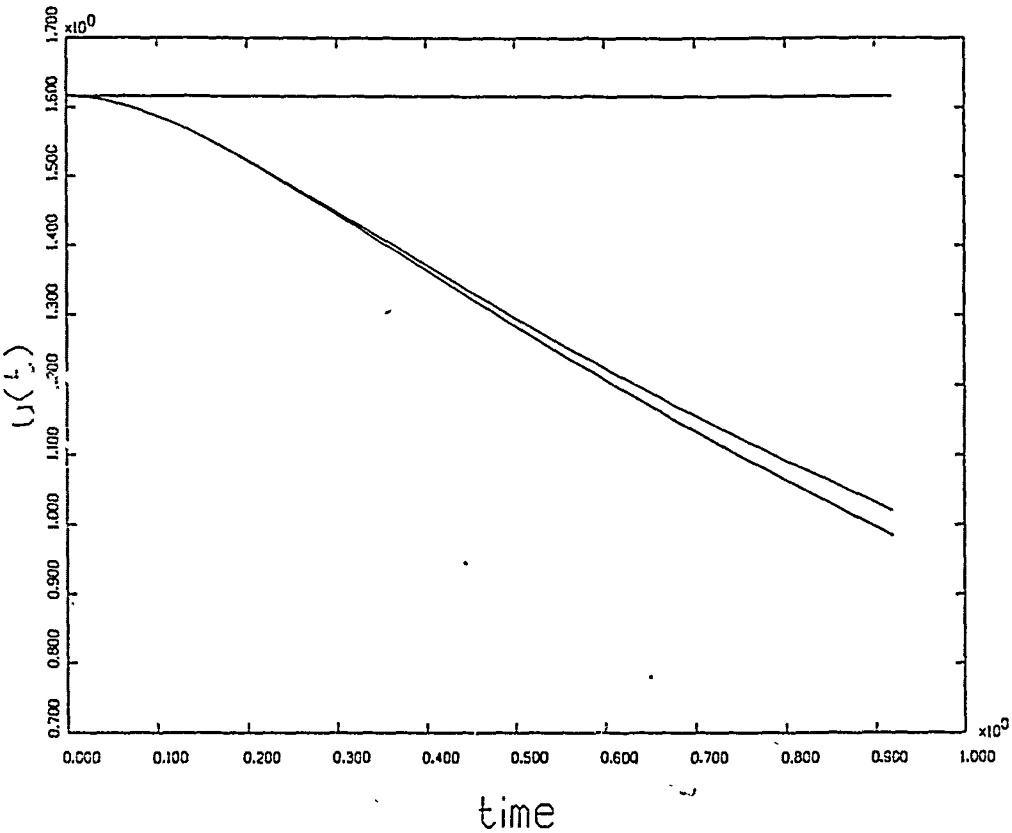


Figure 6. Example 2

### *4.4 Chapter summary*

The nonlinearly damped beam vibration problem has been shown to satisfy the hypotheses of the theory developed in the previous chapter. An example has been presented which demonstrates rapid convergence of the algorithm developed.

# V. *Conclusions and Recommendations*

This chapter provides a brief summary of results which have been demonstrated. It also includes recommendations for further study.

## 5.1 *Conclusions*

A standard mathematical model for approximating the transverse vibration of a beam has been generalized. The existence and uniqueness of solutions have been established. Certain continuous dependence results have been presented.

A nonlinear problem which was not previously known to have a solution, which could be obtained by convergent sequences of approximations, has now been shown to have such a solution.

## 5.2 *Future work*

Several extensions are quite logical for the work contained herein. Certainly it would be desirable to extend the work to more general boundary conditions. This, however, may not be at all straightforward. See Appendix B.

Use of the equations in a parameter identification scheme is certainly appropriate. Indeed, it was with such use in mind that this project was undertaken. This document provides the theoretical basis to undertake a nonlinear identification scheme along the lines of the linear equation based schemes in [12]. The implementation of numerical methods for this problem should be very interesting.

Of course, issues relating to the stability, long term behavior, stability of iterative schemes, and continuous dependence on other elements of the equation are all of interest. This would include time-dependent Trotter-Kato results, such as in [57], [48:pp 17, 49], [32], and [41]. Nonlinear Trotter-Kato type results may be found in several references. See [47:pp 469-476], [41], [50:pp 223-224], [51:pp 403-404], [52:pp 24-25], and [34] for some of the early work. A good summary is in [18]. A product formula version is addressed in [62] and [61].

Application of the style of analysis presented in this work to other equations is also of interest. For example, if the original equation had included rotary inertia, then the nonlinear version of the equation would have been different. A separate analysis is necessary. Allowing time dependence of the fundamental parameters in the derivation is another variation that would be interesting to pursue.

## Appendix A. *Derivation of the Euler-Bernoulli model*

The Euler-Bernoulli model is derived as follows. The total kinetic energy of the beam is assumed to be

$$T = 1/2 \int_0^L m(x) u_t^2(t_0, x) dx. \tag{171}$$

The elastic potential energy due to bending (assumed to be the total potential energy since changes in gravitational potential will be ignored) is assumed to be

$$V = 1/2 \int_0^L EI(x) u_{xx}^2(t_0, x) dx. \tag{172}$$

For convenience, the product $EI(x)$ is treated as a single entity in the equations.

The total mechanical energy in the system for the Euler-Bernoulli model is assumed to be

$$\mathcal{E} = T + V = 1/2 \int_0^L [m(x) u_t^2(t_0, x) + EI(x) u_{xx}^2(t_0, x)] dx \tag{173}$$

and the equations of motion are obtained by setting

$$d\mathcal{E}/dt = 0. \tag{174}$$

Consider the expression for the time rate of change of the total mechanical energy. The arguments of $u$ are suppressed for convenience. Leibnitz rule (see [72:pg 5] or [69:pp 163,170]) is applied to take the differentiation inside the integration.

$$\begin{aligned}
\frac{d\mathcal{E}}{dt} &= \frac{d}{dt}\left(\frac{1}{2}\int_0^L \left[m(x)\left(\frac{\partial u}{\partial t}\right)^2 + EI(x)\left(\frac{\partial^2 u}{\partial x^2}\right)^2\right] dx\right) \\
&= \frac{1}{2}\int_0^L m(x)\frac{\partial}{\partial t}\left(\frac{\partial u}{\partial t}\right)^2 dx + \frac{1}{2}\int_0^L EI(x)\frac{\partial}{\partial t}\left(\frac{\partial^2 u}{\partial x^2}\right)^2 dx \\
&= \int_0^L m(x)\frac{\partial u}{\partial t}\frac{\partial^2 u}{\partial t^2} dx + \int_0^L EI(x)\frac{\partial^2 u}{\partial x^2}\frac{\partial^3 u}{\partial t \partial x^2} dx
\end{aligned} \tag{175}$$

Now expand the second integral in the last expression using integration by parts. (It is assumed that $u$ is smooth enough to justify changes in the order of integration.)

$$\begin{aligned}
\int_0^L EI(x)\frac{\partial^2 u}{\partial x^2}\frac{\partial^3 u}{\partial t \partial x^2}dx &= EI(x)\frac{\partial^2 u}{\partial x^2}\frac{\partial^2 u}{\partial t \partial x}\bigg|_0^L - \int_0^L \frac{\partial^2 u}{\partial t \partial x}\frac{\partial}{\partial x}\left(EI(x)\frac{\partial^2 u}{\partial x^2}\right)dx \\
&= EI(x)\frac{\partial^2 u}{\partial x^2}\frac{\partial^2 u}{\partial t \partial x}\bigg|_0^L - \frac{\partial}{\partial x}\left(EI(x)\frac{\partial^2 u}{\partial x^2}\right)\frac{\partial u}{\partial t}\bigg|_0^L \\
&\quad + \int_0^L \frac{\partial u}{\partial t}\frac{\partial^2}{\partial x^2}\left(EI(x)\frac{\partial^2 u}{\partial x^2}\right)dx \qquad (176)
\end{aligned}$$

Hence,

$$\begin{aligned}
\frac{d\mathcal{E}}{dt} &= \int_0^L \frac{\partial u}{\partial t}\left(m(x)\frac{\partial^2 u}{\partial t^2} + \frac{\partial^2}{\partial x^2}\left(EI(x)\frac{\partial^2 u}{\partial x^2}\right)\right)dx + EI(x)\frac{\partial^2 u}{\partial x^2}\frac{\partial^2 u}{\partial t \partial x}\bigg|_0^L \\
&\quad - \frac{\partial}{\partial x}\left(EI(x)\frac{\partial^2 u}{\partial x^2}\right)\frac{\partial u}{\partial t}\bigg|_0^L \qquad (177)
\end{aligned}$$

It is not difficult to see that for clamped, pinned, or free boundaries the boundary terms in this expression are zero. So, the equations of motion must come from

$$\int_0^L \frac{\partial u}{\partial t}\left(m(x)\frac{\partial^2 u}{\partial t^2} + \frac{\partial^2}{\partial x^2}\left(EI(x)\frac{\partial^2 u}{\partial x^2}\right)\right)dx = 0. \qquad (178)$$

The quantity $\frac{\partial u}{\partial t}$ will not be zero over any interval for any interesting beam. In fact, as time goes by, $\frac{\partial u}{\partial t}$ will take on a wide variety of values. The only way to guarantee $d\mathcal{E}/dt = 0$, is to require

$$m(x)\frac{\partial^2 u}{\partial t^2} + \frac{\partial^2}{\partial x^2}\left(EI(x)\frac{\partial^2 u}{\partial x^2}\right) = 0 \qquad (179)$$

for all $x \in [0, L]$ and $t \geq 0$. This is all made precise in the calculus of variations. The terminology *admissible function* is used for possible values of $\frac{\partial u}{\partial t}$. See, for example [72:pp 217-220].

In the case of constant density, constant cross section, and constant modulus of elasticity, the equation reduces to

$$m\frac{\partial^2 u}{\partial t^2} + EI\frac{\partial^4 u}{\partial x^4} = 0. \tag{180}$$

This is usually written as

$$u_{tt} + (\alpha u_{xx})_{xx} = 0;\ 0 \leq x \leq L;\ 0 \leq t \tag{181}$$

where, consistent with the physical situation, it is assumed that $\alpha > 0$. Since $\alpha$ is a constant the parentheses are not really necessary but they are suggestive of a more general equation.

## Appendix B. *No simple extension to other boundary conditions*

Consider an example which demonstrates that the type of analysis done herein will not carry over to spaces similar to $X$ but with different boundary conditions. In particular, if the boundary conditions on the first component required a zero value on the second derivative but not on the first, then the space would not be complete.

Consider the sequence of functions

$$z_n(x) = \begin{cases} x & 0 \le x \le \frac{1}{2^{n+1}} \\ \frac{2^n}{1-2^n}x^2 - \frac{2^n}{1-2^n}x + \frac{1}{2^{n+2}(1-2^n)} & \frac{1}{2^{n+1}} < x < \frac{2^{n+1}-1}{2^{n+1}} \\ 1-x & \frac{2^{n+1}-1}{2^{n+1}} \le x \le 1 \end{cases} \tag{182}$$

The limit of this sequence is given by

$$z(x) = \lim_{n\to\infty} z_n(x) = \begin{cases} 0 & x = 0 \\ -x^2 + x & 0 < x < 1 \\ 0 & x = 1 \end{cases} \tag{183}$$

It is easy to see that each element of the sequence has second derivative zero at the end points but the limit does not. Slight modifications of this example demonstrate that boundary conditions other than those cited originally do not lead to a complete space in the given norm.

One possible approach to get the second derivative under control would be to base $X$ on $H^3 \times H^0$ instead of $H^2 \times H^0$. But, when a sequence is given as Cauchy in this new $X$ (with the old norm), it is not possible to show that its first components form a Cauchy sequence in $H^3$. So, there is no simple answer here.

Another approach is to let $X$ be the completion of the subset of $H^2 \times H^0$ determined by the boundary conditions. This approach has not been fully analyzed.

Another difficulty arising from other boundary conditions is in establishing the denseness of $D(A)$ in $X$. In particular, Theorem 7 and its generalization in Appendix D will not apply. But, see [12:pp 16-18,42ff] and [58:pg 8, 2.5].

## Appendix C. *Not the generator of an analytic semigroup*

While $X$ has been defined as a real Banach space, complex valued functions could be allowed and the inner product modified (with conjugation of the second factor in the integrand) to make everything proper for the possibility of extension to an analytic semigroup.

**Theorem 75** *The operator $-A_1$ given by (26) is not the generator of an analytic semigroup.*

**Proof:** The proof of this theorem will be given after several preliminaries have been established. □

The key to the argument is Theorem 2.5.2.a,c of [58:pp 61-63]. For convenience the relevant portion of the theorem is stated next. Its proof is in the reference as cited.

**Theorem 76** *Let $S(t)$ be a uniformly bounded $C_0$ semigroup. Let $A$ be the infinitesimal generator of $S(t)$ and assume $0 \in \rho(A)$. The following statements are equivalent:*

a *$S(t)$ can be extended to an analytic semigroup in a sector $\Delta_\delta = \{z : |\arg z| < \delta\}$ and $\|S(z)\|$ is uniformly bounded in every closed subsector $\bar{\Delta}_{\delta'}$, $\delta' < \delta$, of $\Delta_\delta$.*

c *There exist $0 < \delta < \pi/2$ and $M > 0$ such that*

$$\rho(A) \supset \Sigma = \left\{\lambda : |\arg \lambda| < \frac{\pi}{2} + \delta\right\} \cup \{0\} \tag{184}$$

*and*

$$\|R(\lambda : A)\| \leq \frac{M}{|\lambda|} \text{for } \lambda \in \Sigma,\ \lambda \neq 0. \tag{185}$$

**Proof:** See [58:pp 61-63]. □

From Theorem 76 it is clear that the resolvent of $-A$ must include the entire imaginary axis if $-A$ is to be the generator of an analytic semigroup. But, as will be established for $-A_1$ (see (26)), the point spectrum includes infinitely many well spaced points on the imaginary axis. Hence the conclusion.

**Lemma 77** *The point spectrum of $-A_1$ includes arbitrarily large values on the positive and negative imaginary axis.*

**Proof:** First recall that $-A_1$ is skew-adjoint. This leads to the necessity of any eigenvalue being on the imaginary axis. In particular, if $-A_1 u = \lambda u$ then $(-A_1 u, u) = (\lambda u, u)$. But, as was shown earlier, $-A_1$ is dissipative and hence $Re(-A_1 u, u) = -Re(A_1 u, u) \leq 0$. From skew-adjointness, $(-A_1 u, u) = (u, (-A_1)^* u) = (u, A_1 u)$. Thus $Re(A_1 u, u) \leq 0$. Hence, $Re(A_1 u, u) = 0$. On the other hand, $Re(\lambda u, u) = Re(\lambda(u, u)) = Re(\lambda \|u\|_{L_2}^2) = Re(\lambda)\|u\|_{L_2}^2$, which must be zero for arbitrary $u$. Hence $Re(\lambda) = 0$.

This has only established that any eigenvalues for $-A_1$ are constrained to the imaginary axis, it remains to show that there are eigenvalues there. This will be done in a very direct constructive fashion. Solve

$$\begin{pmatrix} 0 & 1 \\ -\alpha D^4 & 0 \end{pmatrix} \begin{pmatrix} u_1 \\ u_2 \end{pmatrix} = \lambda \begin{pmatrix} u_1 \\ u_2 \end{pmatrix} \tag{186}$$

for $\lambda$. This is the same as the system

$$\begin{aligned} u_2 &= \lambda u_1 \\ -\alpha D^4 u_1 &= \lambda u_2. \end{aligned} \tag{187}$$

This can be written, by simple substitution, as

$$-\alpha D^4 u_1 = \lambda^2 u_1 \tag{188}$$

or

$$D^4 u_1 + \frac{\lambda^2}{\alpha} u_1 = 0. \tag{189}$$

It is already known that $\lambda = i\beta$, for some $\beta$, if this equation is to have a nontrivial solution. From the complementary equation

$$r^4 + \frac{\lambda^2}{\alpha} = 0, \tag{190}$$

which can be written

$$r^4 - \frac{\beta^2}{\alpha} = 0, \tag{191}$$

it is clear that the roots are

$$r = \pm\beta^{1/2}\alpha^{-1/4}, \ \pm i\beta^{1/2}\alpha^{-1/4}. \tag{192}$$

For convenience, let $\gamma = \beta^{1/2}\alpha^{-1/4}$. Then

$$\begin{aligned} u &= c_1 e^{\gamma x} + c_2 e^{-\gamma x} + c_3 \cos\gamma x + c_4 \sin\gamma x \\ u' &= \gamma c_1 e^{\gamma x} - \gamma c_2 e^{-\gamma x} - \gamma c_3 \sin\gamma x + \gamma c_4 \cos\gamma x \end{aligned} \tag{193}$$

Now use boundary values to solve for the coefficients.

$$\begin{aligned} u(0) = 0 &\Rightarrow c_1 + c_2 + c_3 = 0 \\ u'(0) = 0 &\Rightarrow c_1 - c_2 + c_4 = 0 \\ u(1) = 0 &\Rightarrow e^{\gamma}c_1 + e^{-\gamma}c_2 + c_3\cos\gamma + c_4\sin\gamma = 0 \\ u'(1) = 0 &\Rightarrow e^{\gamma}c_1 - e^{-\gamma}c_2 - c_3\sin\gamma + c_4\cos\gamma = 0 \end{aligned} \tag{194}$$

The important issue here is to find $\gamma$ so that this system has a nontrivial solution. This requires

$$\begin{vmatrix} 1 & 1 & 1 & 0 \\ 1 & -1 & 0 & 1 \\ e^{\gamma} & e^{-\gamma} & \cos\gamma & \sin\gamma \\ e^{\gamma} & -e^{-\gamma} & -\sin\gamma & \cos\gamma \end{vmatrix} = 0 \tag{195}$$

This is reduced as follows.

$$\begin{vmatrix} 1 & 1 & 1 & 0 \\ 0 & -2 & -1 & 1 \\ 0 & -e^{\gamma} + e^{-\gamma} & -e^{\gamma} + \cos\gamma & \sin\gamma \\ 0 & -e^{\gamma} - e^{-\gamma} & -e^{\gamma} - \sin\gamma & \cos\gamma \end{vmatrix} = 0$$

$$
\begin{vmatrix} -2 & -1 & 1 \\ -e^{\gamma}+e^{-\gamma} & -e^{\gamma}+\cos\gamma & \sin\gamma \\ -e^{\gamma}-e^{-\gamma} & -e^{\gamma}-\sin\gamma & \cos\gamma \end{vmatrix} = 0
$$

$$
\begin{aligned}
-2[-e^{\gamma}\cos\gamma+\cos^2\gamma+e^{\gamma}\sin\gamma+\sin^2\gamma] & \\
+[-e^{\gamma}\cos\gamma+e^{-\gamma}\cos\gamma+e^{\gamma}\sin\gamma+e^{-\gamma}\sin\gamma] & \\
+[e^{2\gamma}+e^{\gamma}\sin\gamma-1-e^{-\gamma}\sin\gamma-e^{2\gamma}+e^{\gamma}\cos\gamma-1+e^{-\gamma}\cos\gamma] &= 0 \\
-4+2e^{\gamma}\cos\gamma+2e^{-\gamma}\cos\gamma &= 0 \\
-1+\frac{e^{\gamma}+e^{-\gamma}}{2}\cos\gamma &= 0 \\
\cosh\gamma\cos\gamma &= 1 \\
\cosh\gamma &= \sec\gamma \qquad (196)
\end{aligned}
$$

The problem has now been reduced to finding real values of $\gamma$ such that $\cosh\gamma = \sec\gamma$. But $\cosh\gamma$ is defined for all real $\gamma$ and is always greater than or equal to 1. On the other hand $\sec\gamma$ has asymptotes at $\gamma = \frac{(2n+1)\pi}{2}$ for $n = \cdots -2, -1, 0, 1, 2, \cdots$. Except for $n = 0$, each asymptote is approached by a unique branch of $\sec\gamma$ which intersects $\cosh\gamma$. (For $n = 0$, two branches intersect at the same point.) Thus an infinite number of arbitrarily large positive and negative values of $\gamma$ are obtained. Now recall $\gamma = \beta^{1/2}\alpha^{-1/4}$, so $\beta = \gamma^2\alpha^{1/2}$ where $\alpha > 0$ is a constant. It is clear that $\beta$ can be arbitrarily large and hence so are the $\lambda$ values.

This establishes the lemma. □

Inconveniently, zero is in the spectrum of $-A$ and hence $0 \notin \rho(-A)$. Hence, an adjustment is necessary to apply Theorem 76. Since $-A$ is the generator of a $C_0$ semigroup its resolvent set contains the positive real axis. A small shift of the problem will put 0 in the resolvent set.

**Lemma 78** *The point spectrum of $-A - \epsilon I$ includes arbitrarily large values on the line $Re(\lambda) = -\epsilon$.*

**Proof:** The argument is exactly the same as in the previous lemma with $\lambda$ replaced by $\lambda + \epsilon$. □

**Lemma 79** *$-A - \epsilon I$ is not the generator of an analytic semigroup.*

**Proof:** By Theorem 6.1 of [19:pg 38], $-A - \epsilon I$ is the generator of a $C_0$ semigroup of contractions. And hence the generator of a uniformly bounded semigroup. (Or, see [30:pg 499].) To justify application of this theorem, note that $-\epsilon I$ is a dissipative operator for any $\epsilon > 0$. In particular

$$\begin{aligned}(\epsilon I u, u) &= -\epsilon(u, u) \\ &= -\epsilon \|u\|^2 \\ &\leq 0 \end{aligned} \tag{197}$$

There is one more preliminary before the proof of the lemma. It must be established that $0 \in \rho(-A - \epsilon I)$. That is, show that $A + \epsilon I$ has a bounded inverse. But, $\epsilon \in \rho(-A) \Rightarrow \epsilon + A$ has a bounded inverse.

Now Theorem 76 applies. Furthermore, any wedge will cross the line $x = -\epsilon$ and hence the sector will include elements of the point spectrum. Thus, the wedge does not lie intirely in the resolvent set and the lemma is established. $\square$

This concludes the preliminaries. The proof of Theorem 75 is structured as a proof by contradiction.

**Proof:** Suppose $-A$ is the generator of an analytic semigroup. Then by Corollary 3.2.2 of [58:pg 81], $-A - \epsilon I$ is the generator of an analytic semigroup. This is a contradiction to the lemma just established.

The theorem follows. $\square$

## Appendix D. *Miscellaneous proofs*

### *D.1 Proof of Lemma 7*

Lemma: If $y \in H^2$ and $y(0) = y(1) = y'(0) = y'(1) = 0$, then $y \in H_0^2$.

**Proof:** When $y \in H^2$, it follows that $y, y', y'' \in L_2$. Since $y'' \in L_2$, there is some sequence $\{y_n''\} \subset C_0^\infty$ such that $y_n'' \to y''$.

Let $y_n'(x) = \int_0^x y_n''(\hat{x})\, d\hat{x}$. Note that

$$\begin{aligned} \left| \int_0^x y_n''(\hat{x})\, d\hat{x} \right| &\leq \int_0^x |y_n''(\hat{x})|\, d\hat{x} \\ &\leq \int_0^1 |y_n''(\hat{x})|\, d\hat{x} \\ &\leq \int_0^1 (y_n''(\hat{x}))^2\, d\hat{x} \end{aligned} \tag{198}$$

which is bounded since $y_n'' \in L_2$. (Note that the last inequality need not hold for $y_n'' < 1$, but in this case boundedness is obvious.) Hence $y_n' \in L_2$.

Claim: $y_n' \to y'$

$$y_n'(x) \to \int_0^x y''(\hat{x})\, d\hat{x} = y'(0) + y'(x) = y'(x). \tag{199}$$

Therefore, the claim holds.

Similarly, let $y_n(x) = \int_0^x y_n'(\hat{x})\, d\hat{x}$, and note that $y_n \in L_2$. It is easy to see that $y_n \in H_0^2$. If $y_n \to y$, then $y \in H_0^2$ by the completeness of $H_0^2$. But this follows by the same line of argument used to show $y_n' \to y'$. □

A more general version of this lemma is also known, as stated below.

Lemma: Let $u \in H^m$ be given. Then $u \in H_0^m$ if and only if $u^{(k)}(0) = u^{(k)}(1) = 0$ for every $k \leq m - 1$.

**Proof:** This is essentially a corollary to Theorem 3.3 of [56:pg 67]. It is taken from [56:pg 91]. The proof requires several other theorems which are also in the cited text. It will not be repeated here. □

### *D.2 Proof of Lemma 24*

Let $(\cdot,\cdot)$ denote an inner product on a linear space $X$. For fixed $x \in X$, let $f(y) = (x,y)$. Then $f$ is continuous.

**Proof:** Let $y_0 \in X$ and $\epsilon > 0$ be given. It is sufficient to identify $\delta > 0$ such that $\|y - y_0\| < \delta \Rightarrow |\,(x,y) - (x,y_0)\,| < \epsilon$. If $x = 0$ the result is trivial. Assume $x \neq 0$. Then

$$\begin{aligned} |\,(x,y) - (x,y_0)\,| &= |\,(x,y-y_0)\,| \\ &\leq \|x\|\,\|y-y_0\|. \end{aligned}$$

(See [54:pg 273], [63:pg 41], or [66:pg 251] for example, for the Schwarz inequality.) But $\|x\|$ is known since $x$ is fixed, so choose $\delta < \epsilon/\|x\|$. □

### *D.3 Comments on the proof of Lemma 42*

The cited proof omits some of the detail concerning the boundedness of $\|\frac{d}{dt}Q\|$. The Uniform Boundedness Principle must be applied. See the proof of Theorem 46 (below) for an example of such an application.

### *D.4 Comments on the proof of Theorem 46*

Theorem: Let $\{A(t)\}_{t\in[0,T]}$ be a stable family of infinitesimal generators of $C_0$ semigroups on $X$. If $D(A(t)) = D$ is independent of $t$ and for $v \in D$, $A(t)v$ is continuously differentiable in $X$ then there exists a unique evolution system $U(t,s)$, $0 \leq s \leq t \leq T$, satisfying the 5 results of Theorem 44 where $Y$ is the set $D$ equipped with the norm $\|v\|_Y = \|v\|_X + \|A(0)v\|_X$.

**Proof:** See [58:pp 145-146]. This appendix is to expand on one portion of the proof cited.

The proof includes a claim that $Q(t) = \lambda_0 I - A(t)$ is an isomorphism of $Y$ onto $X$. It is appropriate to comment on the validation that $Q(t)$ is in fact such an isomorphism. For $\lambda_0 > \omega$, Theorem 1.5.3 of [58:pg 20] gives existence of an inverse for $Q(t)$ which means

it must be 1-1. Theorem 2.16.3 of [22:pg 55] gives onto. (See also [19:pg 13].) Linearity is clear and an algebraic isomorphism is established.

However, for a topological isomorphism, $Q$ and $Q^{-1}$ must each be bounded. This can be a difficult issue. First, consider the boundedness of $Q$.

For any $v \in Y$, $Q(t)v$ is continuous from the hypothesis on $A(t)$. Hence, there is some $M_v$ such that $\|Q(t)v\|_{Y \to X} \leq M_v$ for all $t$. Now the Uniform Boundedness Principle (*eg* [65:pg 196]) applies to give a uniform bound, say $M_Q$ for $\|Q\|$.

Next, recall that bounded, linear, one-to-one, and onto operators have bounded inverses (*eg* [73:pg 70] or [22:pg 47]). Thus, $\|Q^{-1}\|$ is bounded.

Now it is clear that the proposed isomorphism is legitimate.

It is interesting in the above argument that $\|Q\|_{Y \to X}$ is taken to be bounded by hypothesis, yet the norm is specified separately. It is appropriate to verify that these are consistent, *ie*, that $Q$ really is bounded when considered as a mapping from the given $Y$ to the given $X$.

$$\begin{aligned} \|Q(t)\|_{Y \to X} &= \sup_{0 \neq y \in Y} \frac{\|Q(t)y\|_X}{\|y\|_Y} \\ &= \sup_{0 \neq y \in Y} \frac{\|(I + A(t))\,y\|_X}{\|y\|_Y} \\ &= \sup_{0 \neq y \in Y} \frac{\|(I + A(0) - A(0) + A(t))\,y\|_X}{\|y\|_Y} \\ &\leq \sup_{0 \neq y \in Y} \frac{\|(I + A(0))\,y\|_X + \|(A(t) - A(0))\,y\|_X}{\|y\|_Y} \\ &= 1 + \sup_{0 \neq y \in Y} \frac{\|(A(t) - A(0))\,y\|_X}{\|y\|_Y} \end{aligned} \tag{200}$$

But the continuous differentiability of $A(t)$, on a closed and bounded interval, gives Lipschitz continuity to $A$. That is, for some $K$, $\|A(t)y - A(0)y\|_X \leq Kt\|y\|_X$. Then

$$\|Q(t)\|_{Y \to X} \leq 1 + K\hat{T}. \tag{201}$$

*D.5 Proof of Lemma 61*

**Lemma 61** Let $U$ and $V$ be evolution operators. Then

$$V(t,r)y - U(t,r)y = \int_r^t U(t,s)(A^u(s) - A^v(s))V(s,r)y\,ds. \tag{202}$$

**Proof:** The proof consists of taking the derivative of a cleverly chosen quantity and then integrating the result. Recall

$$\begin{aligned} \frac{d}{dt}U(t,s) &= -A(t)U(t,s) \\ \frac{d}{ds}U(t,s) &= U(t,s)A(s). \end{aligned} \tag{203}$$

Now the differentiation is presented.

$$\begin{aligned} \frac{d}{ds}[U(t,s)V(t,s)y] &= \lim_{\Delta s \to 0} \frac{U(t,s+\Delta s)V(s+\Delta s,r)y - U(t,s)V(s,r)y}{\Delta s} \\ &= \lim_{\Delta s \to 0} \frac{U(t,s+\Delta s)V(s+\Delta s,r)y - U(t,s+\Delta s)V(s,r)y}{\Delta s} \\ &\quad + \lim_{\Delta s \to 0} \frac{U(t,s+\Delta s)V(s,r)y - U(t,s)V(s,r)y}{\Delta s} \\ &= \lim_{\Delta s \to 0} U(t,s+\Delta s)\frac{V(s+\Delta s,r) - V(s,r)}{\Delta s}y \\ &\quad + \lim_{\Delta s \to 0} \frac{U(t,s+\Delta s) - U(t,s)}{\Delta s}V(s,r)y \\ &= U(t,s)\frac{d}{ds}V(s,r)y + \frac{d}{ds}U(t,s)V(s,r)y \\ &= U(t,s)(-A^v(s)V(s,r)y) + U(t,s)A^u(s)V(s,r)y \\ &= -U(t,s)A^v(s)V(s,r)y + U(t,s)A^u(s)V(s,r)y \end{aligned}$$

$$= \quad U(t,s)(A^u(s) - A^v(s))V(s,r)y \tag{204}$$

Now, integrate both sides from $r$ to $t$.

$$\begin{aligned} U(t,s)V(t,s)y|_{s=r}^{s=t} &= \int_r^t U(t,s)(A^u(s) - A^v(s))V(s,r)y\,ds \\ U(t,t)V(t,r)y - U(t,r)V(r,r)y &= \int_r^t U(t,s)(A^u(s) - A^v(s))V(s,r)y\,ds \\ V(t,r)y - U(t,r)y &= \int_r^t U(t,s)(A^u(s) - A^v(s))V(s,r)y\,ds \quad \square \end{aligned} \tag{205}$$

A similar argument is available in [53:pg 552].

### *D.6 Proof of Lemma 63*

Lemma: For positive real numbers $a, b,$ and $c$ with $b \geq c$, it follows that $\frac{a+b}{a+c} \leq \frac{b}{c}$.

**Proof:** Let $r_1 = \frac{a+b}{b}$ and $r_2 = \frac{a+c}{c}$. Notice that $r_1 = \frac{a}{b} + 1$ and $r_2 = \frac{a}{c} + 1$, so $b \geq c$ implies $r_2 \geq r_1$ and $\frac{r_2}{r_1} \geq 1$. Then, $r_1 b = a + b$ and $r_2 c = a + c$. Hence

$$\begin{aligned} \frac{a+b}{a+c} &= \frac{r_1 b}{r_2 c} \\ &\leq \frac{r_2}{r_1}\frac{r_1}{r_2}\frac{b}{c} \\ &= \frac{b}{c}. \quad \square \end{aligned} \tag{206}$$

### *D.7 Proofs of Lemmas 70 - 74*

Lemma: For $a, b \geq 0$, $(a^2 + b^2)^{1/2} \leq a + b$.

**Proof:** For positive values of $a$ and $b$ it is clear that

$$\begin{aligned} (a+b)^2 &= a^2 + 2ab + b^2 \\ &\geq a^2 + b^2. \end{aligned} \tag{207}$$

The desired result follows by taking square roots. $\square$

Lemma: For any real numbers $a$ and $b$, $(a+b)^2 \leq 2(a^2+b^2)$.

**Proof:** Note that

$$\begin{aligned} (a-b)^2 &\geq 0 \\ a^2 - 2ab + b^2 &\geq 0 \\ a^2 + b^2 &\geq 2ab. \end{aligned} \tag{208}$$

Now,

$$\begin{aligned} (a+b)^2 &= a^2 + 2ab + b^2 \\ &\leq 2(a^2+b^2). \quad \square \end{aligned} \tag{209}$$

Lemma: For positive real numbers $a, b$, and $c$ it holds that $(a+b+c)^2 \leq 3(a^2+b^2+c^2)$.

**Proof:** This is a straightforward calculation.

$$\begin{aligned} (a+b+c)^2 &= a^2 + 2ab + 2ac + b^2 + 2bc + c^2 \\ &\leq a^2 + (a^2+b^2) + (a^2+c^2) + b^2 + (b^2+c^2) + c^2 \\ &= 3(a^2+b^2+c^2) \quad \square \end{aligned} \tag{210}$$

Lemma: For any positive, real $a, b$, and $c$, it holds that $(a+b+c)^{1/2} \leq (a^{1/2}+b^{1/2}+c^{1/2})$.

**Proof:** This is the same as

$$a+b+c \leq (a^{1/2}+b^{1/2}+c^{1/2})^2. \tag{211}$$

But,

$$\begin{aligned} (a^{1/2}+b^{1/2}+c^{1/2})^2 &= a+b+c+2a^{1/2}b^{1/2}+2a^{1/2}c^{1/2}+2b^{1/2}c^{1/2} \\ &\geq a+b+c \end{aligned} \tag{212}$$

and the desired result follows by taking square roots. □

or

$$\|(I + \lambda A)^{-1}z\| \leq \|z\|. \tag{216}$$

But $\|(I + \lambda A)^{-1}\| = \sup_{\|z\|=1}\{\|(I + \lambda A)^{-1}z\|\}$ and the original form is obtained from the last equation above. For the nonlinear case the expanded version is used for the notion of accretive.

It is also useful, in the nonlinear case, to adopt a set theoretic representation of the operator $A$. That is, for $A \subset X \times X$, define

$$\begin{aligned} Ax &= \{y : [x, y] \in A\} \\ D(A) &= \{x : Ax \neq \phi\} \\ R(A) &= \cup\{Ax : x \in D(A)\} \end{aligned} \tag{217}$$

and note that this allows $A$ to be a multivalued function.

For $A, B \subset X \times X$ and $\lambda \in \Re$ make the natural definitions

$$\begin{aligned} A + B &= \{[x, y + z] : y \in Ax,\ z \in Bx\} \\ \lambda A &= \{[x, \lambda y] : y \in Ax\} \\ A^{-1} &= \{[y, x] : [x, y] \in A\}. \end{aligned} \tag{218}$$

In this setting, call $B \subset X \times X$ accretive if $(I + \lambda B)^{-1}$ is a function for all $\lambda > 0$ and $\|(I + \lambda B)^{-1}x - (I + \lambda B)^{-1}y\| \leq \|x - y\|$ for all $x, y \in D\left((I + \lambda B)^{-1}\right)$.

**Theorem 80 (Crandall-Liggett)** *Let $X$ be a Banach space. Let $A \subset X \times X$ and $\omega \in \Re$, satisfying $A + \omega I$ is accretive, be given. If $R(I + \lambda A) \supset \overline{D(A)}$ for all sufficiently small positive $\lambda$, then*

$$\lim_{n \to \infty} (I + \frac{t}{n}A)^{-n}x \tag{219}$$

*exists for $x \in \overline{D(A)}$ and $t > 0$. Moreover, if $S(t)x$ is defined by this limit, then $S(t) \in Q_\omega(\overline{D(A)})$.*

Note that $A$ is not required to be closed or densely defined or linear or accretive. Also, note that $\omega$ and $A$ are now fixed.

The proof is best approached by first establishing a host of preliminary results. This will, of course, require more notation. Let

$$\begin{aligned} J_\lambda &= (I+\lambda A)^{-1} \\ D_\lambda &= D(J_\lambda) \\ |Ax| &= \inf_{y\in Ax}\{\|y\|\}. \end{aligned} \tag{220}$$

**Lemma 1.2.i:** Choose $\lambda \geq 0$ such that $\lambda\omega < 1$. Then $J_\lambda$ is a function, and for $x, y \in D_\lambda$,

$$\|J_\lambda x - J_\lambda y\| \leq (1-\lambda\omega)^{-1}\|x-y\|. \tag{221}$$

**Proof:** The proof of this lemma begins with a claim.
Claim: $(I + \frac{t}{1+t\omega}A)^{-1}$ has Lipschitz constant $|1+t\omega|$ for $t \geq 0,\ 1+t\omega \neq 0$.

This is shown in the following.

$$\left\|\left(I+\frac{t}{1+t\omega}A\right)^{-1}x - \left(I+\frac{t}{1+t\omega}A\right)^{-1}y\right\| \tag{222}$$

$$\begin{aligned} &= \left\|\left(\frac{1}{1+t\omega}\right)^{-1}\left[((1+t\omega)I+tA)^{-1}x - ((1+t\omega)I+tA)^{-1}y\right]\right\| \\ &= |1+t\omega|\,\|(I+t(\omega I+A))^{-1}x - (I+t(\omega I+A))^{-1}y\| \\ &\leq |1+t\omega|\,\|x-y\| \end{aligned} \tag{223}$$

where the inequality follows by the accretiveness of $A+\omega I$ for $t>0$, and directly for $t=0$, and establishes the claim.

Now, choose $t = \frac{\lambda}{1-\lambda\omega}$ then $\lambda = \frac{t}{1+t\omega}$ where $t \geq 0$. Hence, only those $t$ such that $t\omega > -1$, need be considered. The claim gives $|1+t\omega|$ as a Lipschitz constant for

$(I+\lambda A)^{-1}$, i.e. for $J_\lambda$. But

$$
\begin{aligned}
|1+t\omega| &= 1+t\omega \\
&= \frac{t}{\lambda} \\
&= (1-\lambda\omega)^{-1}
\end{aligned} \tag{224}
$$

and the Lipschitz constant has been validated.

Next, establish that $J_\lambda$ is a function. The key is that $A+\omega I$ is accretive and hence $(I+t(A+\omega I))^{-1}$ is a function for all $t>0$. In particular,

$$
\begin{aligned}
(I+t(A+\omega I))^{-1} &= (I+tA+t\omega I)^{-1} \\
&= ((1+t\omega)I+tA)^{-1} \\
&= \left(\frac{1}{1+t\omega}\right)^{-1}(1+t\omega)^{-1}\,((1+t\omega)I+tA)^{-1} \\
&= (1+t\omega)^{-1}\left(I+\frac{t}{1+t\omega}A\right)^{-1}
\end{aligned} \tag{225}
$$

For $1+t\omega \neq 0$ (legitimate as above) it follows that $(I+\frac{t}{1+t\omega}A)^{-1}$ is a function for all $t>0$. Choose $t$ such that $\lambda = \frac{t}{1+t\omega}$ as above and $J_\lambda$ is a function. This completes the lemma. □

As a corollary to Lemma 1.2.i

$$
\|J_\lambda^n x - J_\lambda^n y\| \leq (1-\lambda\omega)^{-n}\|x-y\| \tag{226}
$$

for all positive integers $n$.

This is established by induction. The lemma has already established the base case. Suppose now that $\|J_\lambda^{n-1}x - J_\lambda^{n-1}y\| \leq (1-\lambda\omega)^{-(n-1)}\|x-y\|$. Let $\hat{x} = J_\lambda^{n-1}x$, $\hat{y} = J_\lambda^{n-1}y$. Then

$$
\begin{aligned}
\|J_\lambda^n x - J_\lambda^n y\| &= \|J_\lambda\hat{x} - J_\lambda\hat{y}\| \\
&\leq (1-\lambda\omega)^{-1}\|\hat{x}-\hat{y}\| \\
&\leq (1-\lambda\omega)^{-n}\|x-y\|
\end{aligned} \tag{227}
$$

and the corollary is established.

Note that, in the above argument, it is assumed that $J_\lambda^k x \in D_\lambda$ for $k = 1, 2, \cdots, n-1$. Also, the fact that $J_\lambda$ is a function was used to show that $J_\lambda x$ is uniquely determined. This can be taken as an additional hypothesis without preventing its desired use later.

As a specific application of the corollary, choose $x = J_\lambda y$. Then

$$\|J_\lambda^{n+1} y - J_\lambda^n y\| \leq (1 - \lambda\omega)^{-n} \|J_\lambda y - y\|. \tag{228}$$

**Lemma 1.2.ii:** Choose $\lambda \geq 0$ such that $\lambda\omega < 1$. Then

$$\|J_\lambda x - x\| \leq \lambda(1 - \lambda\omega)^{-1} |Ax| \tag{229}$$

for $x \in D_\lambda \cap D(A)$.

**Proof:** Let $[x, y] \in A$ be given with $x \in D_\lambda \cap D(A)$. Let $x_1 = (I + \lambda A)^{-1} x$. Then

$$\begin{aligned} (I + \lambda A)x_1 &= x \\ x &= x_1 + \lambda A x_1 \\ &= x_1 + \lambda y_1. \end{aligned} \tag{230}$$

Recall that $J_\lambda x = x_1$ is an alternate notation for $(I + \lambda A)^{-1} x = x_1$. Now

$$\begin{aligned} \|J_\lambda x - x\| &= \|x_1 - x\| \\ &= \|J_\lambda(x_1 + \lambda y_1) - J_\lambda(x + \lambda y)\| \\ &\leq (1 - \lambda\omega)^{-1} \|(x_1 + \lambda y_1) - (x + \lambda y)\| \\ &= \lambda(1 - \lambda\omega)^{-1} \|y\| \end{aligned} \tag{231}$$

which is valid for all $y$. Hence,

$$\begin{aligned} \|J_\lambda x - x\| &\leq \lambda(1 - \lambda\omega)^{-1} \inf_{y \in Ax} \{\|y\|\} \\ &= \lambda(1 - \lambda\omega)^{-1} |Ax| \end{aligned} \tag{232}$$

as desired. Note, in the above arguments, that $(I+\lambda A)x = x + \lambda Ax = x + \lambda y$ means that $x = J_\lambda(x+\lambda y)$ for all $x \in D(A)$. Also, $x_1 + \lambda y_1 - x = 0$. □

**Lemma 1.2.iii:** Choose $\lambda \geq 0$ such that $\lambda\omega < 1$. Let $n$ be a positive integer, $x \in D(J_\lambda^n)$, and $\lambda|\omega| < 1$. Then,

$$\|J_\lambda^n x - x\| \leq n(1-\lambda|\omega|)^{-n+1}\|J_\lambda x - x\|. \tag{233}$$

**Proof:** Observe that

$$\begin{aligned} J_\lambda^n x - x &= J_\lambda^n x - J_\lambda^{n-1}x + J_\lambda^{n-1}x - \cdots + J_\lambda x - x \\ &= \sum_{j=0}^{n-1}\left(J_\lambda^{n-i}x - J_\lambda^{n-(i+1)}\right) \end{aligned} \tag{234}$$

Claim: $x \in D(J_\lambda^n) \Rightarrow x \in D(J_\lambda^{n-1})$.

Since $x \in D(J_\lambda^n)$, $(I+\lambda A)^{-n}x$ is well defined. But, $(I+\lambda A)^{-n}x$ means $[(I+\lambda A)^{-1}x]^n$ and its well definedness implies that of $(I+\lambda A)^{-1}x$. Finally,

$$\begin{aligned} (I+\lambda A)^{-n}x &= (I+\lambda A)^{-(n-1)}(I+\lambda A)^{-1}x \\ &= (I+\lambda A)^{-(n-1)}J_\lambda x \\ &= J_\lambda^{n-1}(J_\lambda x) \end{aligned} \tag{235}$$

and the well definedness of all the other terms gives that of $J_\lambda^{n-1}$. Inductively, the claim holds for all terms of interest. So, all of the newly introduced terms make sense. Thus,

$$\begin{aligned} \|J_\lambda^n x - x\| &= \left\|\sum_{i=0}^{n-1}(J_\lambda^{n-i}x - J_\lambda^{n-(i+1)}x)\right\| \\ &\leq \sum_{i=0}^{n-1}(1-\lambda\omega)^{-n+(i+1)}\|J_\lambda x - x\| \end{aligned} \tag{236}$$

by Lemma 1.2.i.a.

There are three cases to consider. Case 1: $\lambda\omega = 0$. This case is trivial.

Case 2: $\lambda\omega < 0$. It is desired to show that

$$\sum_{i=0}^{n-1}(1-\lambda\omega)^{-n+(i+1)} \leq n(1-\lambda|\omega|)^{-n+1}. \tag{237}$$

But, $\lambda\omega < 0 \Rightarrow 0 < -\lambda\omega$, $1 < 1-\lambda\omega$. Hence, the terms in the summation are increasing. The largest term is $(1-\lambda\omega)^{-n+(n-1+1)} = 1$ and the sum is less than or equal to $n$. This case will be complete if $(1-\lambda|\omega|) \leq 1$ (since then the inverse will be greater than or equal to 1) which the hypothesis $\lambda|\omega| < 1$ clearly assures.

Case 3: $\lambda\omega > 0$. This time

$$\begin{aligned} -\lambda\omega &< 0 \\ 1-\lambda\omega &< 1 \end{aligned} \tag{238}$$

Hence, the largest term in the sum is the first one, $(1-\lambda\omega)^{-n+1}$. The sum is less than or equal to $n(1-\lambda\omega)^{-n+1}$. The proof will be complete if $(1-\lambda\omega)^{-n+1} \leq (1-\lambda|\omega|)^{-n+1}$. But $\lambda\omega > 0 \Rightarrow \omega > 0$, i.e. $|\omega| = \omega$ and the equality holds. □

**Lemma 1.2.iv:** Choose $\lambda \geq 0$ such that $\lambda\omega < 1$. If $\lambda > 0$, $\mu \in \Re$, and $x \in D_\lambda$, then

$$\frac{\mu}{\lambda}x + \frac{\lambda-\mu}{\lambda}J_\lambda x \in D_\mu \tag{239}$$

and

$$J_\lambda x \in J_\mu\left(\frac{\mu}{\lambda}x + \frac{\lambda-\mu}{\lambda}J_\lambda x\right). \tag{240}$$

**Proof:** Since $x \in D_\lambda$, $J_\lambda x$ makes sense. Say

$$\begin{aligned} J_\lambda x &= (I+\lambda A)^{-1}x \\ &= x_0, \end{aligned} \tag{241}$$

whence

$$\begin{aligned} x &= (I+\lambda A)x_0 \\ &= x_0 + \lambda A x_0 \end{aligned}$$

$$= \quad x_0 + \lambda y_0 \tag{242}$$

where $[x_0, y_0] \in A$. Now write

$$\begin{aligned} \frac{\mu}{\lambda}x + \frac{\lambda-\mu}{\lambda}J_\lambda x &= \frac{\mu}{\lambda}(x_0 + \lambda y_0) + \frac{\lambda-\mu}{\lambda}x_0 \\ &= \frac{\mu}{\lambda}x_0 + \mu y_0 + x_0 - \frac{\mu}{\lambda}x_0 \\ &= x_0 + \mu y_0 \end{aligned} \tag{243}$$

It remains to show that $x_0 + \mu y_0 \in D_\mu$. But, $[x_0, y_0] \in A$, so $x_0 \in D(A)$ and $(I + \mu A)x_0 = x_0 + \mu y_0$. This means $(I + \mu A)^{-1}(x_0 + \mu y_0) = x_0$ and the first part of the lemma follows. But, recall $J_\lambda x = x_0$ and $x_0 \in J_\mu(x_0 + \mu y_0)$. Hence, $J_\lambda x \in J_\mu(x_0 + \mu y_0)$ as required. □

As a corollary: If $x \in D(J_\lambda^k)$, then $J_\lambda^k x \in J_\mu\left(\frac{\mu}{\lambda}J_\lambda^{k-1}x + \frac{\lambda-\mu}{\lambda}J_\lambda^k x\right)$ for all integers $k$.

The lemma establishes the base case. It remains to show

$$J_\lambda^k x \in J_\mu\left(\frac{\mu}{\lambda}J_\lambda^{k-1}x + \frac{\lambda-\mu}{\lambda}J_\lambda^k x\right) \Rightarrow J_\lambda^{k+1}x \in J_\mu\left(\frac{\mu}{\lambda}J_\lambda^k x + \frac{\lambda-\mu}{\lambda}J_\lambda^{k+1}x\right) \tag{244}$$

Let $\hat{x} = J_\lambda^k x$. Then, $\hat{x} \in J_\mu\left(\frac{\mu}{\lambda}J_\lambda^{k-1}x + \frac{\lambda-\mu}{\lambda}\hat{x}\right)$ and the desired result is

$$J_\lambda \hat{x} \in J_\mu\left(\frac{\mu}{\lambda}\hat{x} + \frac{\lambda-\mu}{\lambda}J_\lambda \hat{x}\right). \tag{245}$$

Since $\hat{x} \in D(J_\lambda)$, let $J_\lambda \hat{x} = (I + \lambda A)^{-1}\hat{x} = x_1$. Then $\hat{x} = (I + \lambda A)x_1 = x_1 + \lambda A x_1 = x_1 + \lambda y_1$. Note that $[x_1, y_1] \in A$. Write

$$\begin{aligned} \frac{\mu}{\lambda}\hat{x} + \frac{\lambda-\mu}{\lambda}J_\lambda \hat{x} &= \frac{\mu}{\lambda}(x_1 + \lambda y_1) + \frac{\lambda-\mu}{\lambda}x_1 \\ &= x_1 + \mu u_1 \end{aligned} \tag{246}$$

Consider

$$\begin{aligned} (I + \mu A)x_1 &= x_1 + \mu y_1 \\ x_1 &= (I + \mu A)^{-1}(x_1 + \mu y_1) \end{aligned} \tag{247}$$

i.e. $(x_1 + \mu y_1) \in D_\mu$. But, $J_\lambda \hat{x} = x_1$ and $x_1 \in J_\mu(x_1 + \mu y_1)$ so $J_\lambda \hat{x} \in J_\mu(x_1 + \mu y_1) = J_\mu \left( \frac{\mu}{\lambda}\hat{x} + \frac{\lambda-\mu}{\lambda} J_\lambda \hat{x} \right)$.

Before presenting the next lemma, a brief review of some identities among the binomial coefficients is appropriate.

$$\begin{aligned} B(n,m) &= \frac{n!}{m!(n-m)!} \\ B(n,n) &= 1 \\ B(n,0) &= 1 \\ B(n,m) + B(n,m+1) &= B(n+1,m+1) \end{aligned} \tag{248}$$

**Lemma 1.3:** Let $\lambda \geq \mu > 0$, $\lambda\omega < 1$, and $x \in D(J_\lambda^m) \cap D(J_\mu^n)$, where $m$ and $n$ are positive integers with $n \geq m$. Let $\alpha = \frac{\mu}{\lambda}$, $\beta = \frac{\lambda-\mu}{\lambda}$. Then

$$\begin{aligned} \|J_\lambda^n x - J_\mu^m x\| \;\leq\; & (1-\omega\mu)^{-n} \sum_{j=0}^{m-1} \alpha^j \beta^{n-j} B(n,j) \|J_\lambda^{m-j} x - x\| \\ & + \sum_{j=m}^{n} (1-\omega\mu)^{-j} \alpha^m \beta^{j-m} B(j-1,m-1) \|J_\mu^{n-j} x - x\|. \end{aligned} \tag{249}$$

An abbreviation will be useful. For integers $j$ and $k$ satisfying $0 \leq j \leq n$ and $0 \leq k \leq m$, let

$$a_{k,j} = \|J_\mu^j x - J_\lambda^k x\|. \tag{250}$$

Also introduce $\alpha_1 = (1-\mu\omega)^{-1} \frac{\mu}{\lambda}$ and $\beta_1 = (1-\mu\omega)^{-1} \frac{\lambda-\mu}{\lambda}$. In this notation the result of the lemma takes the form

$$a_{m,n} \leq \sum_{j=0}^{m-1} {\alpha_1}^j {\beta_1}^{n-j} B(n,j) a_{m-j,0} + \sum_{j=m}^{n} {\alpha_1}^m {\beta_1}^{j-m} B(j-1,m-1) a_{0,n-j}. \tag{251}$$

**Proof:** The proof begins with a basic inequality relating the $a_{k,j}$. This will require several steps to establish. In the definition of $a_{k,j}$ it is desirable to make the replacement

$$J_\lambda^k x = J_\mu \left( \frac{\mu}{\lambda} J_\lambda^{k-1} x + \frac{\lambda-\mu}{\lambda} J_\lambda^k x \right). \tag{252}$$

This follows from the corollary to Lemma 1.2.iv and Lemma 1.2.i. Now,

$$\begin{aligned}
a_{k,j} &= \|J_\mu^j x - J_\lambda^k x\| \\
&= \left\| J_\mu^j x - J_\mu\left(\frac{\mu}{\lambda}J_\lambda^{k-1}x + \frac{\lambda-\mu}{\lambda}J_\lambda^k x\right)\right\| \\
&\leq (1-\mu\omega)^{-1}\left\| J_\mu^{j-1}x - \left(\frac{\mu}{\lambda}J_\lambda^{k-1}x + \frac{\lambda-\mu}{\lambda}J_\lambda^k x\right)\right\| \\
&\leq (1-\mu\omega)^{-1}\left[\frac{\mu}{\lambda}\|J_\mu^{j-1}x - J_\lambda^{k-1}x\| + \frac{\lambda-\mu}{\lambda}\|J_\mu^{j-1}x - J_\lambda^k x\|\right] \\
&= \alpha_1 a_{k-1,j-1} + \beta_1 a_{k,j-1}
\end{aligned} \tag{253}$$

where the first inequality follows from Lemma 1.2.i, the second inequality uses $\frac{\mu}{\lambda}+\frac{\lambda-\mu}{\lambda}=1$ and the triangle inequality, and the last step is simply a change of notation. This is a basic inequality to be used in the proof.

The proof is by induction on $n$. The inductive proposition $P(n)$ is: For all $m \leq n$

$$a_{m,n} \leq \sum_{j=0}^{m-1} {\alpha_1}^j {\beta_1}^{n-j} B(n,j) a_{m-j,0} + \sum_{j=m}^{n} {\alpha_1}^m {\beta_1}^{j-m} B(j-1,m-1) a_{0,n-j}. \tag{254}$$

Let $n = 1$, for the base case. Note that $m = 1$ is the only possible value for $m$. Claim: The following inequality holds.

$$\begin{aligned}
a_{1,1} &\leq \sum_{j=0}^{0} {\alpha_1}^j {\beta_1}^{1-j} B(1,j) a_{1-j,0} + \sum_{j=1}^{1} {\alpha_1}^1 {\beta_1}^{j-1} B(j-1,0) a_{0,1-j} \\
&= \beta_1 a_{1,0} + \alpha_1 a_{0,0}.
\end{aligned} \tag{255}$$

The basic inequality says $a_{1,1} \leq \alpha_1 a_{0,0} + \beta_1 a_{1,0}$ and the base case is complete. Note that $a_{0,0} = 0$.

Now suppose that

$$a_{m,n} \leq \sum_{j=0}^{m-1} {\alpha_1}^j {\beta_1}^{n-j} B(n,j) a_{m-j,0} + \sum_{j=m}^{n} {\alpha_1}^m {\beta_1}^{j-m} B(j-1,m-1) a_{0,n-j} \tag{256}$$

for all $m \leq n$ and seek to establish

$$a_{m,n+1} \leq \sum_{j=0}^{m-1} \alpha_1{}^j \beta_1{}^{n+1-j} B(n+1,j) a_{m-j,0} + \sum_{j=m}^{n+1} \alpha_1{}^m \beta_1{}^{j-m} B(j-1,m-1) a_{0,n+1-j} \quad (257)$$

for all $m \leq n+1$. Start with the basic inequality,

$$a_{m,n+1} \leq \alpha_1 a_{m-1,n} + \beta_1 a_{m,n}. \quad (258)$$

For each $m \leq n$ the induction hypothesis applies to yield

$$\begin{aligned} a_{m,n+1} \leq\ & \alpha_1 \left( \sum_{j=0}^{m-2} \alpha_1{}^j \beta_1{}^{n-j} B(n,j) a_{m-1-j,0} \right. \\ & \left. + \sum_{j=m-1}^{n} \alpha_1{}^{m-1} \beta_1{}^{j-(m-1)} B(j-1,m-2) a_{0,n-j} \right) \\ & + \beta_1 \left( \sum_{j=0}^{m-1} \alpha_1{}^j \beta_1{}^{n-j} B(n,j) a_{m-j,0} + \sum_{j=m}^{n} \alpha_1{}^m \beta_1{}^{j-m} B(j-1,m-1) a_{0,n-j} \right) \\ =\ & \sum_{j=0}^{m-2} \alpha_1{}^{j+1} \beta_1{}^{n-j} B(n,j) a_{m-(j+1),0} + \sum_{j=m-1}^{n} \alpha_1{}^m \beta_1{}^{j+1-m} B(j-1,m-2) a_{0,n-j} \\ & + \sum_{j=0}^{m-1} \alpha_1{}^j \beta_1{}^{n-j+1} B(n,j) a_{m-j,0} + \sum_{j=m}^{n} \alpha_1{}^m \beta_1{}^{j+1-m} B(j-1,m-1) a_{0,n-j} \end{aligned} \quad (259)$$

Note that $m = n+1$ and $m = 1$ must be treated as separate cases. This will be done later. For now, focus on the first and third summations in the preceeding inequality.

$$\begin{aligned} & \sum_{j=1}^{m-1} \alpha_1{}^j \beta_1{}^{n-j+1} B(n,j-1) a_{m-j,0} + \sum_{j=1}^{m-1} \alpha_1{}^j \beta_1{}^{n-j+1} B(n,j) a_{m-j,0} + \beta_1{}^{n+1} a_{m,0} \\ =\ & \sum_{j=1}^{m-1} \alpha_1{}^j \beta_1{}^{n-j+1} \left[ B(n,j-1) + B(n,j) \right] a_{m-j,0} + \beta_1{}^{n+1} a_{m,0} \\ =\ & \sum_{j=1}^{m-1} \alpha_1{}^j \beta_1{}^{n-j+1} B(n+1,j) a_{m-j,0} + \beta_1{}^{n+1} a_{m,0} \\ =\ & \sum_{j=0}^{m-1} \alpha_1{}^j \beta_1{}^{n-j+1} B(n+1,j) a_{m-j,0} \end{aligned} \quad (260)$$

Next consider the second and fourth summations.

$$\begin{aligned}
& \sum_{j=m}^{n} \alpha_1{}^m \beta_1{}^{j+1-m} \left[B(j-1,m-2)+B(j-1,m-1)\right] a_{0,n-j} + \alpha_1{}^m a_{0,n-(m-1)} \\
= & \sum_{j=m}^{n} \alpha_1{}^m \beta_1{}^{j+1-m} B(j,m-1) a_{0,n-j} + \alpha_1{}^m a_{0,n-(m-1)} \\
= & \sum_{j=m-1}^{n} \alpha_1{}^m \beta_1{}^{j+1-m} B(j,m-1) a_{0,n-j} \\
= & \sum_{j=m}^{n+1} \alpha_1{}^m \beta_1{}^{j-m} B(j-1,m-1) a_{0,n+1-j} \qquad (261)
\end{aligned}$$

It is easy to see, when the terms are recombined, that the desired result is obtained.

Return now to the case $m = 1$. It is necessary to show

$$\begin{aligned}
a_{1,n+1} & \leq \sum_{j=0}^{0} \alpha_1{}^j \beta_1{}^{n-j+1} B(n+1,j) a_{1-j,0} + \sum_{j=1}^{n+1} \alpha_1 \beta_1{}^{j-1} B(j-1,0) a_{0,n+1-j} \\
& = \beta_1{}^{n+1} a_{1,0} + \sum_{j=1}^{n+1} \alpha_1 \beta_1{}^{j-1} a_{0,n+1-j} \qquad (262)
\end{aligned}$$

This must be established for all $n$ so another induction is in order. For $n = 1$, it is required to establish

$$\begin{aligned}
a_{1,2} & \leq \beta_1{}^2 a_{1,0} + \sum_{j=1}^{2} \alpha_1 \beta_1{}^{j-1} a_{0,2-j} \\
& = \beta_1{}^2 a_{1,0} + \alpha_1 a_{0,1} + \alpha_1 \beta_1 a_{0,0}. \qquad (263)
\end{aligned}$$

But, from the basic inequality

$$\begin{aligned}
a_{1,2} & \leq \alpha_1 a_{0,1} + \beta_1 a_{1,1} \\
& = \alpha_1 a_{0,1} + \beta_1 (\alpha_1 a_{0,0} + \beta_1 a_{1,0}) \qquad (264)
\end{aligned}$$

as required. (Note again that $a_{0,0} = 0$.)

Claim: The following holds.

$$a_{1,n+2} \leq \beta_1{}^{n+2} a_{1,0} + \sum_{j=1}^{n+2} \alpha_1 \beta_1{}^{j-1} a_{0,n+2-j} \tag{265}$$

From the basic inequality and the induction hypothesis it follows that

$$\begin{aligned} a_{1,n+2} &\leq \alpha_1 a_{0,n+1} + \beta_1 a_{1,n+1} \\ &\leq \alpha_1 a_{0,n+1} + \beta_1 \left( \beta_1{}^{n+1} a_{1,0} + \sum_{j=1}^{n+1} \alpha_1 \beta_1{}^{j-1} a_{0,n+1-j} \right) \\ &= \alpha_1 a_{0,n+1} + \beta_1{}^{n+2} a_{1,0} + \sum_{j=1}^{n+1} \alpha_1 \beta_1{}^{j} a_{0,n+1-j} \\ &= \beta_1{}^{n+2} a_{1,0} + \sum_{j=0}^{n+1} \alpha_1 \beta_1{}^{j} a_{0,n+1-j} \\ &= \beta_1{}^{n+2} a_{1,0} + \sum_{j=1}^{n+1} \alpha_1 \beta_1{}^{j-1} a_{0,n+2-j} \end{aligned} \tag{266}$$

as required.

Now consider the case $m = n + 1$. This time it is required to show

$$\begin{aligned} a_{n+1,n+1} &\leq \sum_{j=0}^{n} \alpha_1{}^{j} \beta_1{}^{n+1-j} B(n+1,j) a_{n+1-j,0} \\ &\quad + \sum_{j=n+1}^{n+1} \alpha_1{}^{n+1} \beta_1{}^{j-(n+1)} B(j-1,n) a_{0,n+1-j} \\ &= \sum_{j=0}^{n} \alpha_1{}^{j} \beta_1{}^{n+1-j} B(n+1,j) a_{n+1-j,0} + \alpha_1{}^{n+1} a_{0,0}. \end{aligned} \tag{267}$$

This can best be handled by establishing the following intermediate result. Let all conditions be as stated in the Lemma, except, consider $n \leq m$. The result then becomes $a_{m,n} \leq \sum_{j=0}^{n} \alpha_1{}^{j} \beta_1{}^{n-j} B(n,j) a_{m-j,0}$. Notice that once this is established, the desired result follows by replacing $m$ and $n$ with $n+1$ and noting that the $n+1$ term is zero.

With attention on $m \geq n$ it is appropriate to represent $m$ by $n+k$ where $k$ is any appropriate nonnegative integer. Then, proceed with induction on $n$ for the proposition

$$P(n) = \text{For every } k,\ a_{n+k,n} \leq \sum_{j=0}^{n} {\alpha_1}^j {\beta_1}^{n-j} B(n,j) a_{n+k-j,0}. \tag{268}$$

Let $n = 1$. Then

$$a_{1+k,1} \leq \alpha_1 a_{k,0} + \beta_1 a_{k+1,0} \tag{269}$$

by the basic inequality. This clearly has the desired form and the base case is complete.

Suppose $P(n)$ in order to establish $P(n+1)$.

$$\begin{aligned}
a_{n+1+k,n+1} &\leq \alpha_1 a_{n+k,n} + \beta_1 a_{n+1+k,n} \\
&\leq \alpha_1 \sum_{j=0}^{n} {\alpha_1}^j {\beta_1}^{n-j} B(n,j) a_{n+k-j,0} + \beta_1 \sum_{j=0}^{n} {\alpha_1}^j {\beta_1}^{n-j} B(n,j) a_{n+k+1-j,0} \\
&= \sum_{j=0}^{n} {\alpha_1}^{j+1} {\beta_1}^{n-j} B(n,j) a_{n+k-j,0} + \sum_{j=0}^{n} {\alpha_1}^j {\beta_1}^{n+1-j} B(n,j) a_{n+k+1-j,0} \\
&= \sum_{j=1}^{n+1} {\alpha_1}^j {\beta_1}^{n+1-j} B(n,j-1) a_{n+k+1-j,0} + \sum_{j=0}^{n} {\alpha_1}^j {\beta_1}^{n+1-j} B(n,j) a_{n+k+1-j,0} \\
&= {\alpha_1}^{n+1} a_{k,0} + \sum_{j=1}^{n} {\alpha_1}^j {\beta_1}^{n+1-j} [B(n,j-1) + B(n,j)] a_{n+k+1-j,0} \\
&\quad + {\beta_1}^{n+1} a_{n+k+1,0} \\
&= {\alpha_1}^{n+1} a_{k,0} + \sum_{j=1}^{n} {\alpha_1}^j {\beta_1}^{n+1-j} B(n+1,j) a_{n+k+1-j,0} + {\beta_1}^{n+1} a_{n+k+1,0} \\
&= \sum_{j=0}^{n+1} {\alpha_1}^j {\beta_1}^{n+1-j} B(n+1,j) a_{n+k+1-j,0}
\end{aligned} \tag{270}$$

and the induction is complete.

This completes the proof of the lemma. □

Before the next lemma, it is appropriate to recall the Schwartz inequality in the form

$$\sum |x_i y_i| \leq \left(\sum |x_i|^2\right)^{1/2} \left(\sum |y_i|^2\right)^{1/2} \tag{271}$$

where the generic sums suggest that the result holds for both finite and infinite sums, *eg* [54:pg 548].

Several other preliminaries are also in order. The quantity $(\alpha+\beta)^n$ will appear. Write this in its binomial expansion and take derivatives with respect to $\alpha$ to obtain some useful identities.

$$\begin{aligned}
\sum_{j=0}^{n} B(n,j)\alpha^j\beta^{n-j} &= (\alpha+\beta)^n \\
\sum_{j=0}^{n} jB(n,j)\alpha^{j-1}\beta^{n-j} &= n(\alpha+\beta)^{n-1} \\
\sum_{j=0}^{n} jB(n,j)\alpha^j\beta^{n-j} &= \alpha n(\alpha+\beta)^{n-1} \\
\sum_{j=0}^{n} j^2B(n,j)\alpha^{j-1}\beta^{n-j} &= \alpha n(n-1)(\alpha+\beta)^{n-2}+n(\alpha+\beta)^{n-1} \\
\sum_{j=0}^{n} j^2B(n,j)\alpha^j\beta^{n-j} &= \alpha^2 n(n-1)(\alpha+\beta)^{n-2}+\alpha n(\alpha+\beta)^{n-1}
\end{aligned} \tag{272}$$

Recall now, that $\alpha+\beta=1$. The above then simplify to

$$\begin{aligned}
\sum_{j=0}^{n} B(n,j)\alpha^j\beta^{n-j} &= 1 \\
\sum_{j=0}^{n} jB(n,j)\alpha^j\beta^{n-j} &= \alpha n \\
\sum_{j=0}^{n} j^2B(n,j)\alpha^j\beta^{n-j} &= \alpha^2 n(n-1)+\alpha n
\end{aligned} \tag{273}$$

Some similar identities are obtained from the McLaurin series for $(1-\beta)^{-m}$. The series looks like

$$(1-\beta)^{-m} = 1+m\beta+m(m+1)\frac{\beta^2}{2}+\cdots. \tag{274}$$

But

$$B(j-1,m-1) = \frac{(j-1)!}{(m-1)!(j-1-(m-1))!}$$

$$= \frac{(j-1)!}{(m-1)!(j-m)!} \tag{275}$$

and it is clear that

$$\sum_{j=m}^{\infty} B(j-1, m-1)\beta^{j-m} = (1-\beta)^{-m} \tag{276}$$

The next step is to take derivatives with respect to $\beta$ on both sides. This yields

$$\begin{aligned}
\sum_{j=m}^{\infty}(j-m)B(j-1,m-1)\beta^{j-m-1} &= m(1-\beta)^{-m-1} \\
\sum_{j=m}^{\infty}(j-m)B(j-1,m-1)\beta^{j-m} &= m\beta(1-\beta)^{-m-1} \\
\sum_{j=m}^{\infty}(j-m)^2B(j-1,m-1)\beta^{j-m-1} &= m(m+1)\beta(1-\beta)^{-m-2} + m(1-\beta)^{-m-1} \\
\sum_{j=m}^{\infty}(j-m)^2B(j-1,m-1)\beta^{j-m} &= m(m+1)\beta^2(1-\beta)^{-m-2} \\
&\quad +m\beta(1-\beta)^{-m-1}
\end{aligned} \tag{277}$$

Recall that $1-\beta = \alpha$ and reduce these to

$$\begin{aligned}
\sum_{j=m}^{\infty} B(j-1,m-1)\alpha^m\beta^{j-m} &= 1 \\
\sum_{j=m}^{\infty}(j-m)B(j-1,m-1)\alpha^m\beta^{j-m} &= \frac{m\beta}{\alpha} \\
\sum_{j=m}^{\infty}(j-m)^2B(j-1,m-1)\alpha^m\beta^{j-m} &= \frac{m(m+1)\beta^2}{\alpha^2} + \frac{m\beta}{\alpha}
\end{aligned} \tag{278}$$

Some additional modifications to the form of these equations is desired. To prepare for them, note that

$$\begin{aligned}
\alpha + \beta &= 1 \\
m\beta\alpha + m\beta^2 &= m\beta \\
\frac{m\beta}{\alpha} + \frac{m\beta^2}{\alpha^2} &= \frac{m\beta}{\alpha^2}
\end{aligned}$$

$$\frac{m\beta^2}{\alpha^2} = \frac{m\beta}{\alpha^2} - \frac{m\beta}{\alpha}. \tag{279}$$

Now write

$$\begin{aligned}
\sum_{j=m}^{\infty} jB(j-1,m-1)\alpha^m\beta^{j-m} &= \frac{m\beta}{\alpha} + m \\
\sum_{j=m}^{\infty} j^2B(j-1,m-1)\alpha^m\beta^{j-m} &= \frac{m(m+1)\beta^2}{\alpha^2} + \frac{m\beta}{\alpha} + 2m\left(\frac{m\beta}{\alpha} + m\right) - m^2 \\
&= \frac{m(m+1)\beta^2}{\alpha^2} + \frac{m\beta(1+2m)}{\alpha} + m^2 \qquad (280)
\end{aligned}$$

It is finally time to state and prove the lemma for which all these preparations have been made.

**Lemma 1.4:** Let $n \geq m > 0$ be integers and $\alpha$, $\beta$ be positive numbers such that $\alpha + \beta = 1$. Then

$$\sum_{j=0}^{m} B(n,j)\alpha^j\beta^{n-j}(m-j) \leq \left((n\alpha - m)^2 + n\alpha\beta\right)^{1/2} \tag{281}$$

and

$$\sum_{j=m}^{n} B(j-1,m-1)\alpha^m\beta^{j-m}(n-j) \leq \left(\frac{m\beta}{\alpha^2} + \left(\frac{m\beta}{\alpha} + m - n\right)^2\right)^{1/2}. \tag{282}$$

**Proof:** Consider the first inequality. With $n \geq m$, apply the Schwarz inequality to write

$$\begin{aligned}
\sum_{j=0}^{m} B(n,j)\alpha^j\beta^{n-j}(m-j) &\leq \sum_{j=0}^{n} B(n,j)\alpha^j\beta^{n-j}|m-j| \\
&\leq \left(\sum_{j=0}^{n} B(n,j)\alpha^j\beta^{n-j}\right)^{1/2}\left(\sum_{j=0}^{n} B(n,j)\alpha^j\beta^{n-j}(m-j)^2\right)^{1/2} \\
&\leq \left(\sum_{j=0}^{n} B(n,j)\alpha^j\beta^{n-j}\right)^{1/2}\left(\sum_{j=0}^{n} B(n,j)\alpha^j\beta^{n-j}m^2\right. \\
&\qquad \left. - \sum_{j=0}^{n} B(n,j)\alpha^j\beta^{n-j}(2mj) + \sum_{j=0}^{n} B(n,j)\alpha^j\beta^{n-j}j^2\right)^{1/2}
\end{aligned}$$

$$
\begin{aligned}
&= \left(m^2 - 2m\alpha n + \alpha^2 n(n-1) + \alpha n\right)^{1/2} \\
&= \left((n\alpha - m)^2 + \alpha n(1-\alpha)\right)^{1/2} \\
&= \left((n\alpha - m)^2 + \alpha n\beta\right)^{1/2} \qquad (283)
\end{aligned}
$$

This completes the first inequality.

Consider the second inequality.

$$
\begin{aligned}
&\sum_{j=m}^{n} B(j-1, m-1)\alpha^m \beta^{j-m}(n-j) \\
\leq\ &\sum_{j=m}^{\infty} B(j-1, m-1)\alpha^m \beta^{j-m}|n-j| \\
\leq\ &\left(\sum_{j=m}^{\infty} B(j-1, m-1)\alpha^m \beta^{j-m}\right)^{1/2} \left(\sum_{j=m}^{\infty} B(j-1, m-1)\alpha^m \beta^{j-m}(n-j)^2\right)^{1/2} \\
=\ &\left(n^2 - 2n\left(\frac{m\beta}{\alpha} + m\right) + \frac{m(m+1)\beta^2}{\alpha^2} + \frac{m\beta(1+2m)}{\alpha} + m^2\right)^{1/2} \\
=\ &\left(n^2 - \frac{2nm\beta}{\alpha} - 2nm + \frac{m^2\beta^2}{\alpha^2} + \frac{m\beta^2}{\alpha^2} + \frac{m\beta}{\alpha} + \frac{2m^2\beta}{\alpha} + m^2\right)^{1/2} \\
=\ &\left(\frac{m\beta}{\alpha^2} + \frac{m^2\beta^2}{\alpha^2} - \frac{2m\beta}{\alpha}(n-m) + n^2 - 2nm + m^2\right)^{1/2} \\
=\ &\left(\frac{m\beta}{\alpha^2} + \left(\frac{m\beta}{\alpha} - (n-m)\right)^2\right)^{1/2} \qquad (284)
\end{aligned}
$$

This completes the proof of Lemma 1.4. □

Some final preliminaries will now be presented. These are some small details that are best established outside of the main line of argument.

Claim: If $\mu > 0$ and $\mu|\omega| < 1$ then $1 - \mu|\omega| \leq 1 - \mu\omega$.

This is equivalent to $\mu\omega \leq \mu|\omega|$ and is clear. Thus $(1 - \mu|\omega|)^{-1} \geq (1 - \mu\omega)^{-1}$.

Claim: For sufficiently small positive $\lambda$, $(1 - \lambda|\omega|)^{-m} \geq (1 - \lambda|\omega|)^{-(m-j)}$ as long as $j \leq m$.

For small positive values of $\lambda$, it is clear that $1 - \lambda|\omega| \leq 1$ and hence $(1 - \lambda|\omega|)^{-1} \geq 1$. Then the result is clear.

Claim: Given $\epsilon > 0$, there is some positive integer $N$ such that

$$\frac{1}{m} - \frac{1}{n} < \epsilon \text{ whenever } n, m \geq N \tag{285}$$

Consider

$$\begin{aligned} \frac{1}{m} - \frac{1}{n} &< \frac{1}{m} + \frac{1}{n} \\ &\leq \frac{2}{min\{n, m\}} \\ &\leq \frac{2}{N} \end{aligned} \tag{286}$$

Choose $N > \frac{2}{\epsilon}$ and the result is clear.

Claim: If $n$ is a positive integer and $t \in [0, \frac{1}{2}]$, then $(1-t)^{-n} \leq e^{2nt}$.

The proof is by induction. Let $n = 1$. It is required to show

$$(1-t)^{-1} \leq e^{2t} \tag{287}$$

Notice that for $t = 0$ the equality holds. Also note that for $t = \frac{1}{2}$ the inequality holds strictly. This is also true in the form $1 \leq (1-t)e^{2t}$. The base case will be completed by showing that the right hand side is strictly increasing on $t \in [0, \frac{1}{2})$.

Let $f(t) = (1-t)e^{2t}$. Then

$$\begin{aligned} f'(t) &= 2(1-t)e^{2t} - e^{2t} \\ &= (2 - 2t - 1)e^{2t} \\ &= (1-2t)e^{2t} \end{aligned} \tag{288}$$

Since the derivative is always positive, the base case is now clear.

Suppose $(1-t)^{-n} \leq e^{2nt}$, then

$$\begin{aligned} (1-t)^{-(n+1)} &= (1-t)^{-1}(1-t)^{-n} \\ &\leq (1-t)^{-1}e^{2nt} \end{aligned}$$

$$
\begin{aligned}
&\leq e^{2t}e^{2nt} \\
&= e^{2(n+1)t} \qquad (289)
\end{aligned}
$$

and the claim is established.

This concludes the preliminaries and the proof of the theorem will now be presented. The first order of business is to establish the existence of the limit.

Let $x \in D(A)$. Assume $\lambda \geq \mu > 0$, $n \geq m$, $\lambda|\omega| < \frac{1}{2}$. Assume $x \in D(J_\lambda^m) \cap D(J_\mu^n)$. The plan is to establish that $J_{t/n}^n x$ is a Cauchy sequence for rational values of $\mu$ and $\lambda$, then an appeal to continuity will complete the problem. Most of the previous lemmas are used in the algebra.

$$
\begin{aligned}
&\|J_\mu^n x - J_\lambda^m x\| \leq (1-\omega\mu)^{-n}\sum_{j=0}^{m-1}\alpha^j\beta^{n-j}B(n,j)\|J_\lambda^{m-j}x - x\| \\
&+\sum_{j=m}^{n}(1-\omega\mu)^{-j}\alpha^m\beta^{j-m}B(j-1,m-1)\|J_\mu^{n-j}x - x\| \\
\leq\;& (1-\mu|\omega|)^{-n}\sum_{j=0}^{m}\alpha^j\beta^{n-j}B(n,j)\|J_\lambda^{m-j}x - x\| \\
&+\sum_{j=m}^{n}(1-\mu|\omega|)^{-j}\alpha^m\beta^{j-m}B(j-1,m-1)\|J_\mu^{n-j}x - x\| \\
\leq\;& (1-\mu|\omega|)^{-n}\sum_{j=0}^{m}\alpha^j\beta^{n-j}B(n,j)(m-j)(1-\lambda|\omega|)^{-(m-j)+1}\|J_\lambda x - x\| \\
&+\sum_{j=m}^{n}(1-\mu|\omega|)^{-j}\alpha^m\beta^{j-m}B(j-1,m-1)(n-j)(1-\mu|\omega|)^{-(n-j)+1}\|J_\mu x - x\| \\
\leq\;& \Bigg[\lambda(1-\mu|\omega|)^{-n}(1-\lambda|\omega|)^{-m}\sum_{j=0}^{m}\alpha^j\beta^{n-j}B(n,j)(m-j) \\
&+\mu(1-\mu|\omega|)^{-n}\sum_{j=m}^{n}\alpha^m\beta^{j-m}B(j-1,m-1)(n-j)\Bigg]|Ax| \\
\leq\;& \Bigg[\lambda e^{2n\mu|\omega|}e^{2n\lambda|\omega|}\left((n\alpha-m)^2+n\alpha\beta\right)^{1/2} \\
&+\mu e^{2n\mu|\omega|}\left(\frac{m\beta}{\alpha^2}+\left(\frac{m\beta}{\alpha}+m-n\right)^2\right)^{1/2}\Bigg]|Ax|
\end{aligned}
$$

$$
\begin{aligned}
&= \Bigg[\lambda e^{2|\omega|(n\mu+m\lambda)}\left(\left(\frac{n\mu}{\lambda}-m\right)^2+\frac{n\mu}{\lambda}\frac{\lambda-\mu}{\lambda}\right)^{1/2} \\
&\quad +\mu e^{2n\mu|\omega|}\left(\frac{m\lambda^2}{\mu^2}\frac{\lambda-\mu}{\lambda}+\left(\frac{m\lambda}{\mu}\frac{\lambda-\mu}{\lambda}+m-n\right)^2\right)^{1/2}\Bigg]|Ax| \\
&= \Bigg[\lambda e^{2|\omega|(n\mu+m\lambda)}\left(\frac{n^2\mu^2}{\lambda^2}-\frac{2nm\mu}{\lambda}+m^2+\frac{n\mu}{\lambda}-\frac{n\mu^2}{\lambda^2}\right)^{1/2} \\
&\quad +\mu e^{2|\omega|n\mu}\left(\frac{m\lambda^2}{\mu^2}-\frac{m\lambda}{\mu}+\left(\frac{m\lambda}{\mu}-m+m-n\right)^2\right)^{1/2}\Bigg]|Ax| \\
&= \Bigg[\lambda e^{2|\omega|(n\mu+m\lambda)}\left(\left(\frac{n\mu}{\lambda}-m\right)^2+n\mu\left(\frac{1}{\lambda}-\frac{\mu}{\lambda^2}\right)\right)^{1/2} \\
&\quad +e^{2|\omega|n\mu}\left(m\lambda^2-m\mu\lambda+(m\lambda-n\mu)^2\right)^{1/2}\Bigg]|Ax| \\
&= \Bigg[e^{2|\omega|(n\mu+m\lambda)}\left((n\mu-m\lambda)^2+n\mu(\lambda-\mu)\right)^{1/2} \\
&\quad +e^{2|\omega|n\mu}\left(m\lambda(\lambda-\mu)+(m\lambda-n\mu)^2\right)^{1/2}\Bigg]|Ax| \qquad (290)
\end{aligned}
$$

This establishes a slightly tighter bound than the paper gives for its equation 1.9. However, no additional strength is given to the theorem.

For any given $t$ and sufficiently large $n$ and $m$, legitimate values of $\mu$ and $\lambda$ are given by $\frac{t}{n}$, and $\frac{t}{m}$ respectively. Some preparation is needed to substitute this in to the inequality above. The purpose for all this is to establish a result for rational values and then appeal to continuity. The following are useful for substitution.

$$
\begin{aligned}
n\mu = m\lambda &= t \\
n\mu + m\lambda &= 2t \\
n\mu - m\lambda &= 0 \\
&= m\lambda - n\mu \\
n\mu(\lambda-\mu) &= t\left(\frac{t}{m}-\frac{t}{n}\right) \\
&= t^2\left(\frac{1}{m}-\frac{1}{n}\right) \\
&= m\lambda(\lambda-\mu) \qquad (291)
\end{aligned}
$$

Substitution yields

$$
\begin{aligned}
\|J_{t/n}^n x - J_{t/m}^m x\| &\leq \left[e^{2|\omega|(2t)}t\left(\frac{1}{m}-\frac{1}{n}\right)^{1/2} + e^{2|\omega|t}t\left(\frac{1}{m}-\frac{1}{n}\right)^{1/2}\right]|Ax| \\
&= \left[e^{4t|\omega|}t\left(\frac{1}{m}-\frac{1}{n}\right)^{1/2} + e^{2|\omega|t}t\left(\frac{1}{m}-\frac{1}{n}\right)^{1/2}\right]|Ax| \\
&\leq 2te^{4t|\omega|}\left(\frac{1}{m}-\frac{1}{n}\right)^{1/2}|Ax| \\
&\leq e^{2|\omega|}\left(\frac{1}{m}-\frac{1}{n}\right)^{1/2}|Ax| \qquad (292)
\end{aligned}
$$

Recall that $\omega$ is fixed (from the very beginning) and that consideration is currently being given to a particular $x$. The limit

$$\lim_{n\to\infty} J_{t/n}^n x \qquad (293)$$

exists since the terms have just been shown to form a Cauchy sequence in a Banach space.

From the corollary to Lemma 1.2.i

$$\|J_{t/n}^n x - J_{t/n}^n y\| \leq \left(1 - \frac{t}{n}\omega\right)^{-n} \|x - y\| \qquad (294)$$

Recall that

$$\lim_{n\to\infty}\left(1 - \frac{\omega t}{n}\right)^{-n} = e^{\omega t}. \qquad (295)$$

Let

$$S(t)x = \lim_{n\to\infty} J_{t/n}^n x. \qquad (296)$$

Then $S(t)$ has $e^{\omega t}$ as a Lipschitz constant. Thus, if $S$ is a semigroup, then $S \in Q_\omega$.

The definition of $S$ for elements of $D(A)$ has been given. But $S$ must be defined on all of $X$. The extension will rely on the denseness of $D(A)$ in $X$. Let $x \in \overline{D(A)}\backslash D(A)$ be given. There is a sequence $\{x_i\}$ of elements in $D(A)$ which converges to $x$. The issue to be resolved is the existence of

$$\lim_{n\to\infty} J_{t/n}^n x. \qquad (297)$$

Let $\epsilon > 0$ be given. Choose $N_1$ such that $i > N_1$ implies $\|x_i - x\| < \frac{\epsilon}{4e^{\omega t}}$. Choose $N_2$ such that for any $i, j > N_1$ and $n > N_2$ it follows that

$$\|J^n_{t/n}x_i - J^n_{t/n}x_j\| \leq 2e^{\omega t}\|x_i - x_j\|. \tag{298}$$

Let $N = max\{N_1, N_2\}$. Then

$$\|J^n_{t/n}x_i - J^n_{t/n}x_j\| < \epsilon \tag{299}$$

and the sequence is Cauchy as desired. Also,

$$\begin{aligned} \|J^n_{t/n}x_i - J^n_{t/n}x\| &= \|J^n_{t/n}x_i - \lim_{j\to\infty} J^n_{t/n}x_j\| \\ &\leq 2e^{\omega t}\|x_i - x_j\| \\ &= 2e^{\omega t}\|x_i - x + x - x_j\| \\ &< 2e^{\omega t}\frac{\epsilon}{2e^{\omega t}} \\ &= \epsilon. \end{aligned} \tag{300}$$

Now it makes sense to define

$$S(t)x = \lim_{n\to\infty}\lim_{i\to\infty} J^n_{t/n}x_i. \tag{301}$$

The Lipschitz continuity of $S(t)$ will now be established. Let $\tau > t \geq 0$ be given. For $x \in D(A)$, legitimate values of $\mu$ and $\lambda$ are given by $\frac{t}{n}$ and $\frac{\tau}{n}$ respectively. Choose $n = m$ in equation 1.9 and write

$$\begin{aligned} \|J^n_{\tau/n}x - J^n_{t/n}x\| &\leq \left[\left((t-\tau)^2 + t\frac{\tau - t}{n}\right)^{1/2} e^{2|\omega|(t+\tau)} \right. \\ &\quad \left. + \left(\tau\frac{\tau - t}{n} + (\tau - t)^2\right)^{1/2} e^{2|\omega|t}\right] |Ax| \\ \lim_{n\to\infty}\|J^n_{\tau/n}x - J^n_{t/n}x\| &\leq \left[|t-\tau|e^{2|\omega|(t+\tau)} + (\tau - t)e^{2|\omega|t}\right] |Ax| \\ &= (\tau - t)|Ax|\left(e^{2|\omega|(t+\tau)} + e^{2|\omega|t}\right) \end{aligned} \tag{302}$$

But,

$$\begin{aligned}\lim_{n\to\infty} \|J^n_{\tau/n}x - J^n_{t/n}x\| &= \|\lim_{n\to\infty} J^n_{t/n}x - \lim_{n\to\infty} J^n_{\tau/n}x\| \\ &= \|S(\tau)x - S(t)x\| \end{aligned} \tag{303}$$

Thus $S(t)x$ is Lipschitz continuous in $t$ for $x \in D(A)$ and bounded $t$ intervals.

Claim: For $x \in \overline{D(A)}$, $S(t)x$ is continuous as a function of $t$.

Let $x \in \overline{D(A)}\backslash D(A)$ be given. There is a sequence $\{x_i\} \to x$ where each $x_i$ is in the domain of $A$. Let $\epsilon > 0$ be given. It is required to find some $\delta > 0$ such that $\|S(t)x - S(\tau)x\| < \epsilon$ whenever $\|\tau - t\| < \delta$. But, for any $x_i$

$$\|S(\tau)x_i - S(t)x_i\| \leq |t-\tau|\,|Ax|\left(e^{2|\omega|(t+\tau)} + e^{2|\omega|t}\right) \tag{304}$$

Furthermore, $\|S(t)x_i - S(t)x\| < \epsilon/3$ for sufficiently large $i$. Then

$$\begin{aligned}\|S(\tau)x - S(t)x\| &= \|S(\tau)x - S(\tau)x_i + S(\tau)x_i - S(t)x_i + S(t)x_i - S(t)x\| \\ &\leq \|S(\tau)x - S(\tau)x_i\| + \|S(\tau)x_i - S(t)x_i\| + \|S(t)x_i - S(t)x\| \\ &\leq \epsilon/3 + |t-\tau|\,|Ax|\left(e^{2|\omega|(t+\tau)} + e^{2|\omega|t}\right) + \epsilon/3. \end{aligned} \tag{305}$$

All that remains is to choose $\delta$ such that

$$|t-\tau|\,|Ax|\left(e^{2|\omega|(t+\tau)} + e^{2|\omega|t}\right) < \epsilon/3 \tag{306}$$

That is, it is required that

$$\delta\,|Ax|\left(e^{2|\omega|2\hat{t}} + e^{2|\omega|t}\right) < \epsilon/3 \tag{307}$$

where $\hat{t}$ is the maximum allowable value for $t$.

Choose

$$\delta = \frac{\epsilon}{4|Ax|\left(e^{4|\omega|\hat{t}} + e^{2|\omega|t}\right)} \tag{308}$$

and the claim follows.

It is clear, from the definition of the $J_\lambda$ terms, that $S(0)$ is the identity. The previous claim, applied for $t = 0$ gives the continuity required for $S \in Q_\omega$. The proof is concluded with the establishment of $S(t+\tau) = S(t)S(\tau)$. As has been the pattern, there are a few preliminaries. Namely, by applying the definition and previously established continuity results, it follows that

$$\begin{aligned}
[S(t)]^m &= \left[\lim_{n\to\infty} J^n_{t/n}\right]^m \\
&= \lim_{n\to\infty} \left[J^n_{t/n}\right]^m \\
&= \lim_{n\to\infty} \left[J^m_{t/n}\right]^n
\end{aligned} \tag{309}$$

Also, it now follows that

$$\begin{aligned}
S(mt) &= \lim_{n\to\infty} J^n_{mt/n} \\
&= \lim_{k\to\infty} J^{mk}_{mt/mk} \\
&= \lim_{k\to\infty} \left[J^k_{t/k}\right]^m \\
&= [S(t)]^m
\end{aligned} \tag{310}$$

It will now be established that the semigroup property holds for rational values of $t$ and $\tau$. Let $l$, $k$, $r$, and $s$ be positive integers. Then

$$\begin{aligned}
S\left(\frac{l}{k}+\frac{r}{s}\right) &= S\left(\frac{ls+rk}{ks}\right) \\
&= \left[S\left(\frac{1}{ks}\right)\right]^{ls+rk} \\
&= \left[S\left(\frac{1}{ks}\right)\right]^{ls}\left[S\left(\frac{1}{ks}\right)\right]^{rk} \\
&= S\left(\frac{l}{k}\right)S\left(\frac{r}{s}\right)
\end{aligned} \tag{311}$$

But, since $S(t)$ is continuous as a function of $t$ and Lipschitz as an operator on $X$, it follows that the relationship holds for all real numbers. □

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